Math 170A-5 Yeliussizov. Midterm 1

Exam time: 4:00-4:50 PM, Jan 31, 2018

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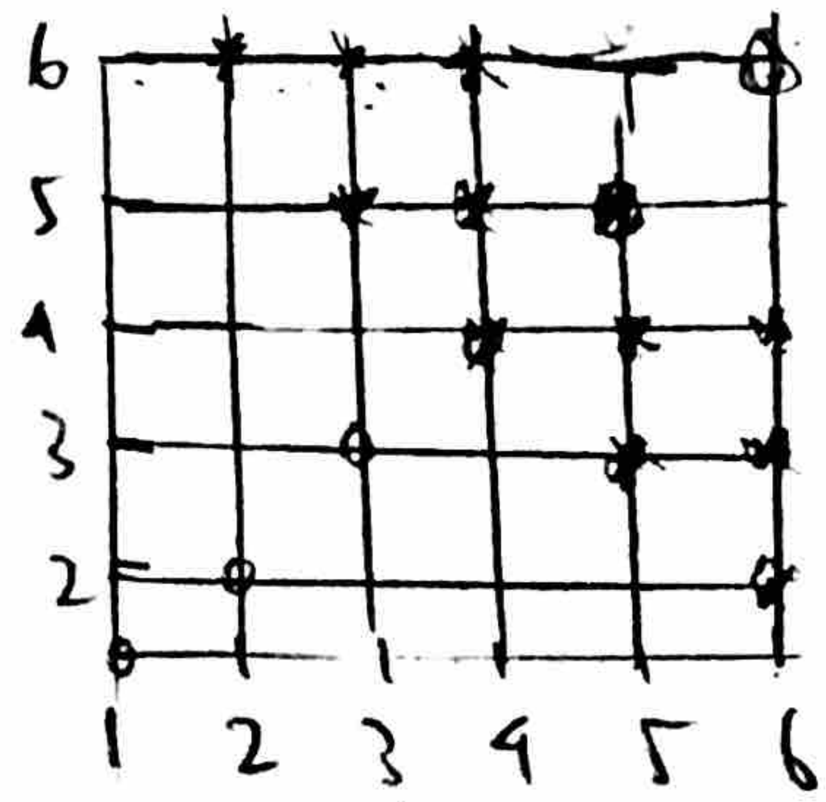
There are 5 problems.
No books, notes, calculators, phones, conversations, etc.
Turn off your cell phones.

P 1 (8)	P 2 (8)	P 3 (8)	P 4 (8)	P 5 (8)	Total (40 pt)
8	8	8	8	6	38

Problem 1. (8 points) Roll two fair six-sided dice.
(a) (4 points) Find the probability that both outcomes are equal or their sum is at least 8 and at most 10.
(b) (4 points) Find the probability of the same event as in (a) given that at least one outcome is 6.

a)



$P(\text{equal}) = \frac{6}{36}$

$P(\text{at least 8 and at most 10}) = \frac{12}{36}$

$P(\text{equal} \cap \text{at least 8 and at most 10}) = \frac{2}{36}$

4

$P(\text{equal} \cup \text{sum at least 8 and at most 10}) = P(\text{equal}) + P(\text{at least 8 and at most 10}) - P(\text{equal} \cap \text{at least 8 and at most 10})$

$= \frac{6}{36} + \frac{12}{36} - \frac{2}{36} = \frac{16}{36} = \frac{4}{9}$

b) E = qual
A = at least 8 and at most 10
B = P(E ∪ A)
C = at least one outcome is 6

We want $P(B|C) = \frac{P(B \cap C)}{P(C)}$ $\quad P(B \cap C) = \frac{7}{36}$

$P(C) = \frac{11}{36}$

4

$P(B|C) = \frac{7/36}{11/36} = \frac{7}{11}$

Problem 2. (8 points) There are m boxes and n of them contain a prize, where $3 < n \leq m$. You randomly choose three boxes.

(a) (4 points) Find the probability that each of the three boxes contains a prize.

(b) (4 points) Find the probability that at least one of them contains a prize.

a) denominator - $\binom{m}{3}$ choosing 3 boxes from all boxes

numerator = $\binom{n}{3}$ choosing 3 boxes w/ prizes from all boxes w/ prizes

$= \dfrac{\binom{n}{3}}{\binom{m}{3}}$

4

b) A = at least one contains prize

A^c = none contain prize

$P(A^c) = \dfrac{\binom{m-n}{3}}{\binom{m}{3}}$ ← choose 3 from boxes that do not contain a prize

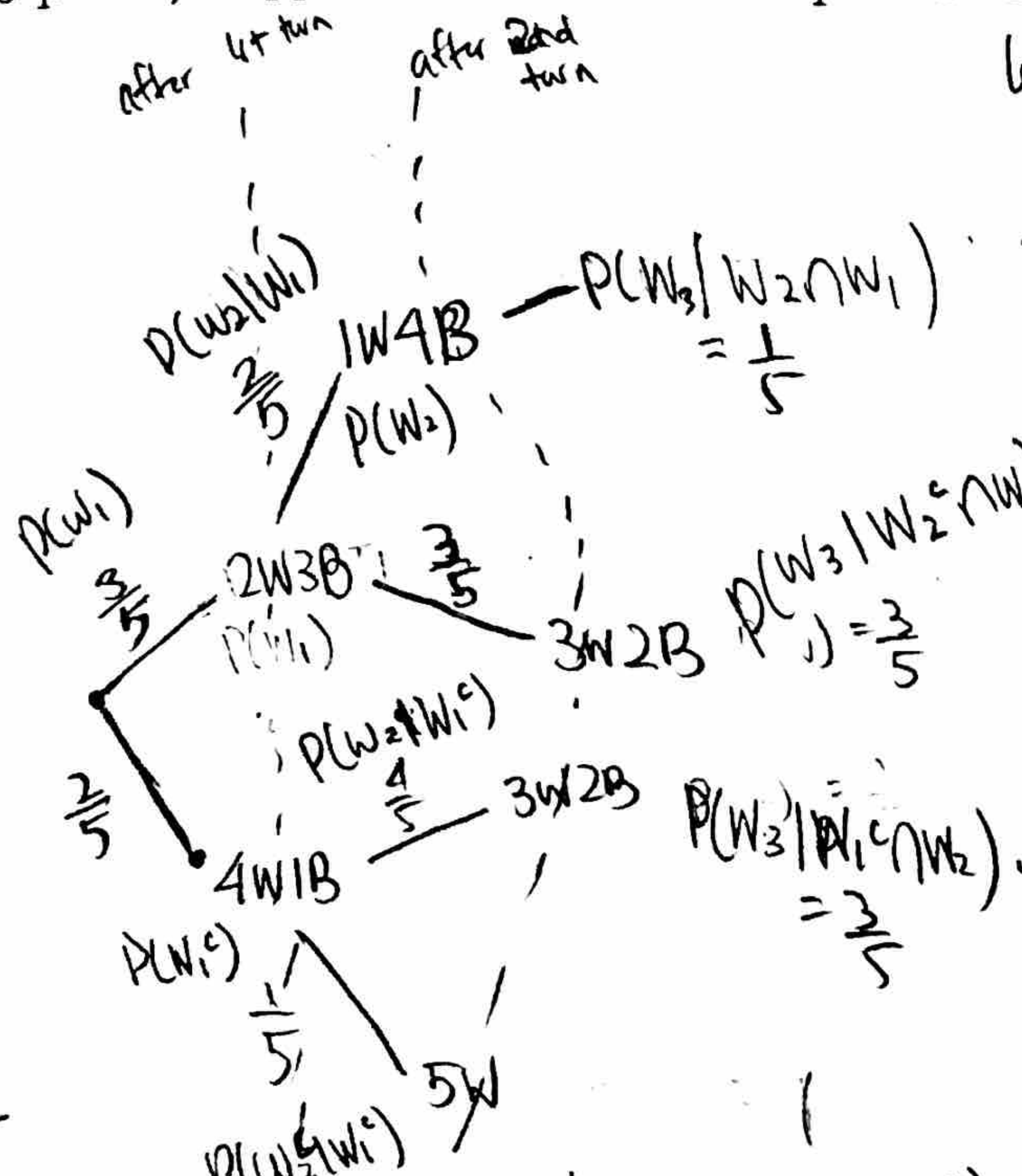
$P(A) = 1 - P(A^c)$

$= 1 - \dfrac{\binom{m-n}{3}}{\binom{m}{3}}$

4

Problem 3. (8 points) Consider a box with 3 white and 2 black balls. You repeat the following **twice**: take a random ball from the box and replace it by a ball of opposite color (i.e., exchange randomly chosen white to black, or black to white). After these exchanges, you choose a ball from the box.
(a) (5 points) Find the probability that it is a white ball.
(b) (3 points) Suppose a white ball was picked. Find the probability that the box contains only white balls.

a)



let $P(W_i)$ be the probability of picking white for the ith turn where each turn is taking a random box and replacing it w/ ball of opposite color.

From tree and total prob

$$P(W_3) = P(W_2)P(W_3|W_2) + P(W_2^c)P(W_3|W_2^c)$$

where $P(W_2) = P(W_1)P(W_2|W_1) + P(W_1^c)P(W_2|W_1^c)$

$P(W_2^c) = P(W_1)P(W_2^c|W_1) + P(W_1^c)P(W_2^c|W_1^c)$

$P(W_1) = \frac{3}{5}$

$P(W_1^c) = \frac{2}{5}$

5

$$P(W_3) = \frac{3}{5}\cdot\frac{2}{5}\cdot\frac{1}{5} + \frac{3}{5}\cdot\frac{3}{5}\cdot\frac{3}{5} + \frac{2}{5}\cdot\frac{4}{5}\cdot\frac{3}{5} + \frac{2}{5}\cdot\frac{1}{5}\cdot 1$$

$$= \frac{6}{125} + \frac{27}{125} + \frac{24}{125} + \frac{2}{125} = \frac{6}{125}+\frac{27}{125}+\frac{24}{125}+\frac{10}{125} = \frac{67}{125}$$

b) box w/ 5W

$$P(5W|W_3) = \frac{P(W_3|5W)P(5W)}{P(W_3)} = \frac{1\cdot\left(\frac{2}{5}\cdot\frac{1}{5}\right)}{69/125} = \frac{2/125}{67/125} = \frac{10/125}{69/125} = \frac{10}{67}$$

3

Problem 4. (8 points)
(a) (2 points) Write down the definition of independence of three events.
(b) (6 points) Suppose that A, B, C are independent events. Show that $A \cup B$ and C are independent as well.

a) 3 events A, B, C are independant if all of the following are satisfied

1) $P(A \cap B) = P(A)P(B)$ 3) $P(B \cap C) = P(B)P(C)$
2) $P(A \cap C) = P(A)P(C)$ 4) $P(A \cap B \cap C) = P(A)P(B)P(C)$

b) $A \cup B$ and C are independent if

$$\underline{P((A \cup B) \cap C) = P(A \cup B)P(C)}$$

RHS: $P((A \cup B) \cap C) = P((A \cap C) \cup P(B \cap C))$

from property that $P(X \cup Y) = P(X) + P(Y) - P(X \cup Y)$ $\longrightarrow$ $= P(A \cap C) + P(B \cap C) - P((A \cap C) \cap (B \cap C))$

from above: if A, B, C independent $= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$

$= P(A)P(C) + P(B)P(C) - P(A)P(B)P(C)$

$= P(C)(P(A) + P(B) - P(A)P(B))$ if independant $= P(A \cap B)$

$= P(C)(P(A) + P(B) - P(A \cap B))$

$= P(C)P(A \cup B)$

RHS = LHS

$\therefore$ $A \cup B$ and C are independent if A, B, C are independent

Problem 5. (8 points) Toss a fair coin n times. What is the probability that the number of heads is larger than the number of tails?

Let h = number of heads
Let t = number of tails

$p(h>t) = p(h>\lfloor \frac{n}{2} \rfloor)$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

all possibilities where greater than $\lfloor \frac{n}{2} \rfloor$ heads

$$= \sum_{k=\lfloor \frac{n}{2} \rfloor+1}^{n} \binom{n}{k}\left(\frac{1}{2}\right)^{k}\left(\frac{1}{2}\right)^{n-k}$$

$$= \sum_{k=\lfloor \frac{n}{2} \rfloor+1}^{n} \binom{n}{k}\left(\frac{1}{2}\right)^{n}$$

why?

if n = odd $= \frac{1}{2}$

if n = even $= \dfrac{\sum_{k=\lfloor \frac{n}{2} \rfloor+1}^{n} \binom{n}{k}}{2^n}$

~~$1 + 3 = \frac{4}{8}$~~

~~$4 \cdot \frac{1}{16} = \frac{5}{16}$~~

~~$\frac{1}{32} + \frac{5}{32} + \frac{10}{32} = \frac{16}{32}$~~