1. Suppose a fair die is rolled 10 times. Find the numerical values of the expectations of each of the following random variables :

a). the sum of the smallest 2 numbers in the first 3 rolls. [5 points]

b). the number of different faces that appear in the 10 rolls. [5 points]

a) Let S = sum of 2 smallest numbers in first 3 rolls. , X_i be number of ith roll

$S = X_1 + X_2 + X_3 - \max\{X_1, X_2, X_3\}$ Let $\max(X_1, X_2, X_3) = M$.

For $y = 1, 2, \dots 6$. $P(X_i \le 1) = \frac{1}{6}$ $P(X_i \le 2) = \frac{2}{6}$ ... $P(X_i \le 6) = 1$, $k = 1, 2, 3 \dots 6$.

$P(M = y) = P(M \le y) - P(M \le y-1) = P(X_1 \le y) P(X_2 \le y) P(X_3 \le y) - P(X_1 \le y-1) P(X_2 \le y-1) P(X_3 \le y-1)$

$= \left(\frac{y}{6}\right)^3 - \left(\frac{y-1}{6}\right)^3$

$ES = EX_1 + EX_2 + EX_3 - EM$

$EX_1 = EX_2 = EX_3 =$, $EX_i = \frac{1}{6}\times 1 + \frac{1}{6}\times 2 + \dots + \frac{1}{6}\times 6 = 3.5$ 3

$ES = 3.5 \times 3 - \sum_{k=1}^{6} k \left(\frac{k}{6}\right)^3 - \left(\frac{k-1}{6}\right)^3$

b) Let $Z_i = 1$ if ith face appear. 0 otherwise.

$Z = Z_1 + \dots + Z_6$. $E(Z) = E(Z_1 + \dots + Z_6) = 6EZ_1$ by symmetry.

$EZ_1 = P(Z_1 = 1) = 1 - P(Z_1 = 0) = 1 - P(\text{face } i \text{ does not appear in 10 rolls})$

$= 1 - \left(\frac{5}{6}\right)^{10}$

$EZ = 6EZ_1 = 6 \times \left[1 - \left(\frac{5}{6}\right)^{10}\right]$

2. Let A_1, A_2, and A_3 be events with probabilities 1/5, 1/4 and 1/3 respectively. Let N be the number of these events that occur.

a). Find the expectation $\mathbb{E}N$. [3 points]

In each of the following case, find the variance Var N:

b). A_1, A_2 and A_3 are disjoint. [3 points]

c). A_1, A_2 and A_3 are independent. [3 points]

d). $A_1 \subset A_2 \subset A_3$. [4 points]

a) Let $I_{A_i} = 1$ if A_i occur, 0 otherwise.

$p_i = \mathbb{P}(I_{A_i}=1) = \mathbb{P}(A_i)$ 3

$\mathbb{E}N = \mathbb{E}(I_{A_1}+I_{A_2}+I_{A_3}) = \mathbb{E}(I_{A_1}) + \mathbb{E}(I_{A_2}) + \mathbb{E}(I_{A_3}) = \frac{1}{5}+\frac{1}{4}+\frac{1}{3} = \frac{12+20+15}{60} = \frac{47}{60}.$

b) if $A_1 \cdot A_2 \cdot A_3$ are disjoint:

$\mathbb{E}N^2 = \sum_{i=1}^{3} \mathbb{E}(N^2 | A_i)\mathbb{P}(A_i) = \Sigma$

c) if $A_1 \cdot A_2 \cdot A_3$ are independent:

$\mathbb{E}N^2 = 1^2 \mathbb{P}(\text{one occur}) + 2^2 \mathbb{P}(2 \text{ occur}) + 3^2 (3 \text{ occur})$

$= 1\times(\frac{1}{5}+\frac{1}{4}+\frac{1}{3}) + 4\times(\frac{1}{5}\times\frac{1}{4}+\frac{1}{4}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{5}) + 9\times(\frac{1}{5}\times\frac{1}{4}\times\frac{1}{3})$

$\text{Var}N = \mathbb{E}N^2 - (\mathbb{E}N)^2$ - - - 0

$= 4\times(\frac{1}{5}\times\frac{1}{4}+\frac{1}{4}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{5}) + \cancel{9\times(\frac{1}{5}\times\frac{1}{4}\times\frac{1}{3})}$

3. A deck of 100 cards numbered from 1 to 100 is well shuffled. After the shuffle, there is a match at place k if the k^{th} card from the top of the deck is the card with number k. Let X = # of matches in the deck.

a). Find the probability $\mathbb{P}(X = 0)$. [4 points]

b). Find the expectation $\mathbb{E}X$. [4 points]

c). Find the variance VarX. [4 points]

Let $I_k = 1$ if k^{th} card is a match. 0 if otherwise.

a) $\mathbb{P}(X=0)$ - every card is a mismatch.

$\mathbb{P}$(ith card is a mismatch) $= \frac{99}{100}$ by symmetry.

$\mathbb{P}(X=0) = \left(\frac{99}{100}\right)^{100}$

b) $\mathbb{E}X = \sum_{k=1}^{100} I_k$ $\quad \mathbb{E}(I_k) = \mathbb{P}(k\text{th is a match}) = \frac{1}{100}$. by symmetry

$\mathbb{E}X = \sum_{k=1}^{100} I_k = 100 \cdot \mathbb{E}I_k = 100 \times \frac{1}{100} = 1.$

c) $\operatorname{Var}X = \mathbb{E}X^2 - (\mathbb{E}X)^2$

$\mathbb{E}X^2 = \sum_{x=1}^{100} x^2 P_x$

$P_x = \mathbb{P}(X=x) = \mathbb{P}(x \text{ matches in deck}) = \left(\frac{1}{100}\right)^x \left(\frac{99}{100}\right)^{100-x}$

$\mathbb{E}X^2 = \sum_{x=1}^{100} x^2 \cdot \left(\frac{1}{100}\right)^x \left(\frac{99}{100}\right)^{100-x}$

$\operatorname{Var}X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \sum_{x=1}^{100} x^2 \left(\frac{1}{100}\right)^x \left(\frac{99}{100}\right)^{100-x} - 1^2.$

4. Pick letters with replacement from the following box until you get the letter 'M'. Let N = # of letters drawn and X = # of times the letter 'A' is drawn.

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a). Find the probability $\mathbb{P}(N = 10)$. [3 points]

b). Find the conditional probability $\mathbb{P}(X = 3 \mid N = 10)$. [3 points]

c). Find the probability $\mathbb{P}(X = x)$ for $x = 0, 1, 2, \ldots$ Do you recognize this distribution? [4 points]

a) $\mathbb{P}(\text{get } M) = \frac{2}{11}$, $N = \text{Geo}(\frac{2}{11})$

with $p = \frac{2}{11}$, number of trials before success (get M) is $\mathbb{P}(X=k) = (1-p)^{k-1}p$.

Plug in $p = \frac{2}{11}$, $k = 10$. $\mathbb{P}(N=10) = (1-\frac{2}{11})^9 \times \frac{2}{11}$

b) first 9 are not M, $\mathbb{P}(\text{get } A) = \frac{2}{9}$

$X = \text{Bin}(9, \frac{2}{9})$ By binomial factor, $\mathbb{P}(X=3 \mid N=10) = \binom{9}{3}(\frac{2}{9})^3(\frac{7}{9})^6$.

Since this is essentially choosing 3 positions out of 9 slots. the 3 slots have possibility $\frac{2}{9}$ of being A. other 6 slots has $(\frac{7}{9})$ possibility of not being A.

c) $\mathbb{P}(X=x) = \sum_{k=x+1}^{\infty} \mathbb{P}(X=x \mid N=k)\,\mathbb{P}(N=k)$

$$= \sum_{k=x+1}^{\infty} \binom{k-1}{x}\left(\frac{2}{9}\right)^x\left(\frac{7}{9}\right)^{k-1-x} \cdot \left(\frac{9}{11}\right)^{k-1} \times \left(\frac{2}{11}\right)$$

$$= \left(\frac{2}{9}\right)^x\left(\frac{9}{11}\right)^x\left(\frac{2}{11}\right)\sum_{k=x+1}^{\infty}\frac{(k-1)\cdots(k-x)}{x!}\left(\frac{7}{9}\right)^{k-1-x}\left(\frac{9}{11}\right)^{k-1-x}$$

$$= \left(\frac{2}{11}\right)^{x+1}\sum_{k=x+1}^{\infty}\frac{(k-1)\cdots(k-x)}{x!}\left(\frac{7}{11}\right)^{k-1-x}$$

$$= \left(\frac{2}{11}\right)^{x+1}\sum_{k=x+1}^{\infty}\frac{1}{x!}\left(\frac{d^x(r^{k-1})}{dr^x}\Big|_{r=\frac{7}{11}}\right)$$

$$= \left(\frac{2}{11}\right)^{x+1}\frac{d^x\left(\sum_{k=x+1} r^{k-1}\right)}{dr^x}\Big|_{r=\frac{7}{11}}$$

$$= \left(\frac{2}{11}\right)^{x+1}\cdot\frac{1}{x!}\cdot\frac{d^x\frac{1}{1-r}}{dr^x}\Big|_{r=\frac{7}{11}} = \left(\frac{2}{11}\right)^{x+1}\cdot\frac{1}{x!}\cdot\frac{x!}{(1-\frac{7}{11})^{x+1}}$$

$$= \left(\frac{2}{11}\right)^{x+1}\cdot\left(\frac{11}{4}\right)^{x+1} = \left(\frac{1}{2}\right)^{x+1} = \left(\frac{1}{2}\right)^x\frac{1}{2}$$

is a Geometric distribution

5. Suppose that $X \sim \text{Pois}(2)$, $Y \sim \text{Geo}(1/3)$ on $\{0, 1, 2, \ldots\}$, and X is independent of Y. Find the probability $\mathbb{P}(Y \geq X)$. [7 points]

$Y = \text{Geo}(\frac{1}{3})$ $\mathbb{E}(Y) = y \cdot P_y =$ $(1-\frac{1}{3})^k \cdot \frac{1}{3}.$

$X = \text{Pois}(2)$ $\mathbb{E}(X=k) =$

6. On average, 0.5% of the purchasers of airline tickets do not appear for the departure of their flight. How many tickets should be sold for a flight on an airplane which has 200 seats, such that with probability 0.95 everybody who appears at the departure will have a seat? [8 points]