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Math 170A: Probability

Second Midterm - version 1

- Answer the questions in the spaces provided on the question sheets.
- If you run out of room for an answer, continue on the back of the page and indicate clearly that you're doing this.
- The last page is intentionally left blank for scratch work, and you may tear it out of the booklet.
- There are no notes, calculators, or aids of any kind allowed.
- Be sure to present your solution fully and clearly. In particular, if you are using a theorem, law, or property, please state this.
- The questions are not in order of difficulty, nor are they arranged by topic.
- On some questions, the parts may be graded independently of each other. For example, if you make an error in part (a) and it propogates to part (b), you may lose points for (b) as well.

I have read and understood the above directions. [redacted]

Signature: [redacted]

Question	Points	Score
1	20	18
2	28	22
3	28	27
4	24	20
Total:	100	87

1. Let X be a discrete random variable.

   (a) (5 points) Give the definition of the *probability mass function* of X.

   The probability mass function of X gives the probability of an outcome of an experiment that can be mapped from Ω, where $\omega \in \Omega$ and $X(\omega) = x$

   4 almost- need to be more precise.

   (b) (5 points) Give the definition of the *expectation* of X.

   The expectation is the mean or average of X weighted by its probabilities.

   4

   (c) (5 points) Let A be an event with $P(A) > 0$. Give the definition of the *conditional expectation* of X given A.

   The conditional expectation is the expectation of X computed with respect to event A.

   $$\mathbb{E}[X \mid A] = \sum_{x=1}^{\infty} x \frac{P_X(\{X=x\} \cap A)}{P(A)}$$

   5

   (d) (5 points) Let Y be another random variable. Define what it means for X and Y to be *independent*.

   If X & Y are independent, then the outcomes of X & Y do not affect each other. Thus, the pmf's of X Y combined (joint) is the same as the sum of pmf of X + pmf of Y. Also if computing conditional pmf (ex. $P_{X|Y}$), then if X, Y independent

   $$P_{X|Y}(x \mid y) = P_X(x)$$

   5

2. A fair, six sided die is thrown. Let X be the outcome of the roll. You place a bet: if the outcome of the throw is the number 2, you gain \$5; if it is any number other than 2 you lose \$1.

   (a) (7 points) Write down the probability mass function of X.

$X = (\{ i \})$
$\forall\ i = 1, 2, \ldots, 6$

$$P_X(X) = \begin{cases} 5 & x = 2 \\ -1 & x = 1, 3, 4, 5, 6 \\ 0 & \text{o.w.} \end{cases}$$

+1

   (b) (7 points) Compute the expected outcome of the bet.

$$\mathbb{E}[X] = \sum_{x=1}^{6} x\, P_X(X=x)$$
$$= 1/6\,[5(1) + (-1)(5)]$$
$$= 1/6\,[5-5] = \emptyset$$

   (c) (7 points) Compute the variance of the outcome of the bet.

$$\operatorname{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$
$$= 1/6\,[5^2(1) + (-1)^2(5)] - \emptyset^2$$
$$= 1/6\,(25+5) - \emptyset = 30/6$$
$$= 5$$

   (d) (7 points) Compute the expected outcome of the bet, given that the outcome of the throw is **not** the number 3.

$$\mathbb{E}[X \mid X \neq 3] = \sum_{x=1}^{6} x\, \frac{P_X(\{X=x\} \cap \{X \neq 3\})}{P(\{X \neq 3\})}$$
$$= \frac{5(1)(1/6)}{(5/6)} + \frac{(-1)(4)(1/6)}{(5/6)}$$
$$= 1 - 4/5 = 1/5$$

3. A coin shows heads on a toss with probability p, independent of other tosses. The coin is tossed n times. For $i = 1, 2, ..., n$, let X_i take the value 1 if the i-th toss is heads, and 0 if the i-th toss is tails. Let Y be given by,

$$Y = \sum_{i=1}^{n} X_i. \quad \rightarrow \text{ Y is Binomial}$$

(a) (8 points) Find the expectation of X_i.

$$P_X(X_i) = \begin{cases} p & x=1 \\ 1-p & x=0 \end{cases}$$

$$\mathbb{E}[X_i] = \sum_{x=0}^{1} x \, P_X(X_i = x)$$
$$= 1(p) + 0(1-p) = p$$

(b) (8 points) Find the variance of X_i.

$$\text{var}(X_i) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$
$$= (1)^2 p + 0^2(1-p) - p^2$$
$$= p - p^2 = p(1-p)$$

(c) (6 points) Find the probability mass function of Y.

Y is sum of X_i outcomes, so Y counts the number of heads.

$$P_Y(y) = \binom{n}{i} p^{i} (1-p)^{n-i}$$

for $Y = \ldots$

(d) (6 points) Find the expectation of Y.

$$\mathbb{E}[Y] = \sum_{y=1}^{n} y \, P_Y(\{Y = y\})$$

↳ but since $Y = \sum_{i=1}^{n} X_i$ & X_i's are all independent of each other.

$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i]$$
$$= X_1 + X_2 + \cdots + X_n = p + p + \cdots + p$$
$$= np.$$

4. A coin shows heads on a toss with probability p, independent of other tosses. The coin is tossed until it shows tails, after which point it is not tossed again. Let X be the total number of tosses, including the last one.

↳ geometric !

(a) (8 points) Find the expectation of X.

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k \, P_X(\{X=k\})$$
$$= \sum_{k=1}^{\infty} k p (1-p)^{k-1}$$
$$= \sum_{k=1}^{\infty} \underbrace{p}_{a} \underbrace{k(1-p)^{k-1}}_{x r^{x-1}}$$
$$= p/(1-(1-p))^2 = p/p^2$$
$$= 1/p$$

$P_X(x) = p(1-p)^{k-1}$ for k-tosses. $\forall k = 1, 2, \ldots$

geometric:
$$\sum_{x=1}^{\infty} a x r^{x-1}$$
↑ 1st derivative
$$= a/(1-r)^2$$

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almost - here tails is a "success"

(b) (8 points) For each toss (including the final one), a die is thrown. Compute the expected value of the sum of the faces of the dice.

Let Y be the sum of the die rolls.

5 ✓

First we find expectation of 1 roll.
$$\mathbb{E}[Y \mid X=1] = \sum_{y=1}^{6} y \, P_{Y|X}(y \mid X=1)$$
$$= 1 \cdot 1/6 + 2 \cdot 1/6 + \ldots + 6 \cdot 1/6 = 3.5$$

Then $\mathbb{E}[Y \mid X_i] = 3.5 i$ (since X_i tosses/rolls are independent)

not quite

$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_{i=1}^{\infty} Y \mid X_i\right] = \sum_{i=1}^{\infty} \mathbb{E}[Y \mid X_i] = \sum_{i=1}^{\infty} 3.5 i$$

(c) (8 points) For each toss (including the final one), a die is thrown. Let Y be the sum of the faces of the dice. Find the expectation of X, given that $Y = 1$.

$$\mathbb{E}[X \mid Y=1] = \sum_{k=1}^{\infty} k \, P_{X|Y}(\{X=k\} \mid \{Y=1\})$$
$$= \sum_{k=1}^{\infty} k \frac{P_{X,Y}(\{X=x\}, \{Y=1\})}{P(\{Y=1\})}$$
$$= (1)(1) = 1$$

8

↳ $Y=1$ only when $X=1$ since each toss adds at least 1 to sum.

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$$\mathbb{E}[Y \mid X_i] = 3.5i$$

$$\mathbb{E}[Y] = \sum_{y=1}^{\infty} \mathbb{E}[Y \mid X_i] P(X_i)$$

$$= \sum_{y=1}^{\infty} 3.5p(1-p)^{y-1}$$

$$=$$