Name: [redacted]

UCLA ID number: [redacted]

Math 170A: Probability

First Midterm - version 1

- Answer the questions in the spaces provided on the question sheets.
- If you run out of room for an answer, continue on the back of the page and indicate clearly that you're doing this.
- The last page is intentionally left blank for scratch work, and you may tear it out of the booklet.
- There are no notes, calculators, or aids of any kind allowed.
- Be sure to present your solution fully and clearly. In particular, if you are using a theorem, law, or property, please state this.
- The questions are not in order of difficulty, nor are they arranged by topic.

I have read and understood the above directions.

Signature: [redacted]

Question	Points	Score
1	24	24
2	22	22
3	30	30
4	24	22
Total:	100	98

1. A fair coin is flipped three times. All tosses are independent.

(a) (8 points) What is the probability that Heads comes up exactly once?

$\Omega = \{HHH, HTH, HTT, HHT, TTT, THT, THH, TTH\}$

$|\Omega| = 8$

Let H = event that Heads occurs exactly once

$P(H) = \frac{|H|}{|\Omega|} = \frac{3}{8}$

$H = \{HTT, THT, TTH\}$

(b) (8 points) What is the probability that Heads comes up more often than Tails?

Using same Ω from (a). $|\Omega| = 8$

Let A = event that Heads occurs more than Tails

$A = \{HHH, HTH, HHT, THH\}$

$P(A) = \frac{|A|}{|\Omega|} = \frac{4}{8} = \frac{1}{2}$

(c) (8 points) Given that the first coin comes up Tails, what is the probability that Heads comes up more often than Tails?

Using same Ω from (a).

Let T = event that first flip is Tails

A = event that Heads occurs more than Tails.

definition of conditional probability ↳

$$P(A \mid T) = \frac{P(A \cap T)}{P(T)} = \frac{1/8}{1/2} = \frac{1}{8}\left(\frac{2}{1}\right) = \boxed{1/4}$$

$T = \{TTT, THT, THH, TTH\}$ $|T| = 4$

$P(T) = 4/8 = 1/2$

$A = \{HHH, HTH, HHT, THH\}$

$A \cap T = \{THH\}$

$P(A \cap T) = 1/8$

2. **You don't have to simplify your final answer. For example, an answer of the form $25 + \frac{39}{0.7}$ is fine.**

Apex Health Company designed a test for a disease. The disease occurs in 2% of the population. If a person has the disease, the test comes up positive 90% of the time, and if a person does not have the disease, the test comes up positive 10% of the time.

A person is picked at random from the population. Let A be the event that the person has the disease. The test is performed on this person. Let B be the event that the test is positive.

(a) (6 points) Find $P(A)$, $P(B|A)$, and $P(B|A^c)$.

$$\left.\begin{aligned} P(A) &= 0.02 \\ P(B|A) &= 0.9 \\ P(B|A^c) &= 0.1 \end{aligned}\right\} \text{given in problem statement.}$$

(b) (8 points) Find $P(B)$.

Using Total Probability Law

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$$

$$= 0.9(0.02) + (0.1)(1-0.02)$$

$$P(B) = 0.9(0.02) + (0.1)(0.98)$$

Continuation. Here is the text of the question, repeated verbtim:

Apex Health Company designed a test for a disease. The disease occurs in 2% of the population. If a person has the disease, the test comes up positive 90% of the time, and if a person does not have the disease, the test comes up positive 10% of the time.

A person is picked at random from the population. Let A be the event that the person has the disease. The test is performed on this person. Let B be the event that the test is positive.

(c) (8 points) Given that the test result is positive, what is the probability that the person has the disease?

Given B, probability A?

$P(A|B) = ?$

using Baye's theorem.

Note $A \cup A^c = \Omega$ so Ω is partitioned into A & A^c.

$$P(A|B) = \frac{P(B|A)P(A)}{\sum \text{ conditioning on } A}$$

From (a)

$P(B|A) = 0.9$

$P(A) = 0.02$

$P(B|A^c) = 0.1$

$$= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

$$= \frac{0.9(0.02)}{0.9(0.02) + 0.1(1-0.02)}$$

$$P(A|B) = \frac{0.9(0.02)}{0.9(0.02) + 0.1(0.98)}$$

3. Two fair, 6-sided dice are rolled. The rolls are independent. Let D be the event that the sum of the numbers on the two dice is 11, E the event that the sum of the numbers on the two dice is 7, and F the event that the first die shows 1.

For (a) through (e)

$\Omega = \{(i,j) \mid i,j \in \{1,2,\dots,6\}\}$ where i = 1st die, j = 2nd die

$|\Omega| = 36$

(a) (6 points) Find the probability that F occurs.

$F = \{(i,j) \mid i = 1, j \in \{1,\dots,6\}\}$

$|F| = 6$

$P(F) = \frac{|F|}{|\Omega|} = \frac{6}{36} = 1/6$

(b) (6 points) Find the probability that E occurs.

$E = \{(i, 7-i) \mid i \in \{1,2,3,\dots,6\}\}$

$|E| = 6$

$P(E) = \frac{|E|}{|\Omega|} = \frac{6}{36} = 1/6$

(c) (6 points) Find the probability that D occurs.

$D = \{(i, 11-i) \mid i \in \{1,2,3,\dots 6\}$ and $(11-i) \in \{1,2,\dots,6\}\}$

$|D| = 2$

$P(D) = \frac{|D|}{|\Omega|} = \frac{2}{36} = 1/18$

(d) (6 points) Are F and D independent?

F & D independent if $P(F \cap D) = P(F)P(D)$

$P(F \cap D) = 0$ (if 1st die is 1, sum can never = 11)

$P(F)P(D) = 1/6\,(1/18)$

$P(F \cap D) \neq P(F)P(D)$

$\therefore$ F & D are NOT independent.

(e) (6 points) Are F and E independent?

F & E independent if $P(F \cap E) = P(F)P(E)$

$F \cap E = \{(1,6)\}$

$P(F \cap E) = 1/36$

$P(F)P(E) = 1/6 \cdot 1/6 = 1/36$

$P(F \cap E) = P(F)P(E)$

$\therefore$ F & E are independent.

4. There is data on families in the United States, including how many children are in each family, the age of the child, the gender of each child, and the day of the week that each child was born. You may assume that a child is equally likely to be a girl or a boy, is equally likely to be born on any of the days of the week, and that the day of birth and gender of a child are independent of each other and of other children.

   (a) (7 points) Consider the collection of families which have exactly two children. A family is picked at random from this collection. What is the probability that the family has exactly two boys?

BB BG
GB GG

Ω = families w/ 2 kids

$$|\Omega| = \underbrace{\binom{2}{1}}_{\text{1st kid gender}} \underbrace{\binom{2}{1}}_{\text{2nd gender}} \underbrace{\binom{7}{1}}_{\text{day of week 1st}} \underbrace{\binom{7}{1}}_{\text{day of week 2nd}}$$

$$= \binom{2}{1}^2 \binom{7}{1}^2 = 2^2 \cdot 7^2$$

B = family has 2 boys ?

$$|B| = 1 \cdot 1 \cdot \binom{7}{1}\binom{7}{1} = \binom{7}{1}^2$$

↳ set 1st kid & 2nd kid's gender.

$$P(B) = \binom{7}{1}^2 \Big/ \binom{2}{1}^2\binom{7}{1}^2 = \frac{1}{\binom{2}{1}^2} = \frac{1}{4}$$ ✓

   (b) (7 points) Consider the collection of families with exactly two children, one of whom is a boy. A family is picked at random from this collection. What is the probability that the family has exactly two boys?

→ at least 1 boy.

Ω = families 2 kids, one boy

BG GB BB

$$|\Omega| = \binom{2}{1} \cdot 1 \cdot \binom{7}{1} \cdot \binom{7}{1} + 1 \cdot \binom{7}{1} \cdot \binom{7}{1}$$

↳ set one kids gender & since at least also add in set both gender

$$= \binom{2}{1}\binom{7}{1}^2 + \binom{7}{1}^2 = 3\binom{7}{1}^2$$

B = family has 2 boys

$$|B| = 1 \cdot 1 \cdot \binom{7}{1} \cdot \binom{7}{1} = \binom{7}{1}^2$$

$$P(B) = \frac{\cancel{\binom{7}{1}^2}}{(3)\cancel{\binom{7}{1}^2}} = \frac{1}{(3)} = \frac{1}{3}$$ ✓

(Continuation of Question 4.)

(c) (7 points) Consider the collection of families with exactly two children in which the older child is a boy. A family is chosen at random from this collection. What is the probability that the family has exactly two boys?

BG BB

②

Ω = families 2 kids, oldest boy

↳ set oldest childs gender.

$$|\Omega| = 1 \cdot \binom{2}{1} \cdot \binom{7}{1} \cdot \binom{7}{1}$$

$$= 2\binom{7}{1}^2 = 2 \cdot 7^2$$

B = families has 2 boys

$$|B| = 1 \cdot 1 \cdot \binom{7}{1} \cdot \binom{7}{1} = \binom{7}{1}^2$$

$$P(B) = \cancel{\binom{7}{1}^2} \Big/ 2\cancel{\binom{7}{1}^2} = 1/2$$

✓

(d) (3 points) Consider the collection of families with exactly two children in which one of the children is a boy born on a Friday. A family is chosen at random from this collection. What is the probability that the family has exactly two boys?

F F F F

BG GB

BB GG ②

F F F F

Ω = families 2 kids, one born Friday

$$|\Omega| = \binom{2}{1} \cdot \binom{2}{1} \cdot \binom{7}{1} \cdot 1$$

↑ set a childs bday on Friday.

$$= \binom{2}{1}^2 \binom{7}{1}$$

BB

F F ②

B = family has 2 boys

set birthday to a Friday because Ω requires one to have a bday on friday.

$$|B| = 1 \cdot 1 \cdot \binom{7}{1} \cdot \binom{1}{1} = \binom{7}{1}$$

here you're counting 2nd kid w/ F b-day, but it could be either!

$$P(B) = \frac{\cancel{\binom{7}{1}}}{\binom{2}{1}^2 \cancel{\binom{7}{1}}} = \frac{1}{\binom{2}{1}^2} = \frac{1}{4} = 1/4$$

1

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