Math 170A, Fall 2015
Practice Midterm 1

Last name	
First name	
Student ID	

Use the provided space for your solutions (you may use the additional page at the end). Show your work. Write only one solution per problem - problems with multiple solutions will not be graded. Numeric expressions can be left unsimplified. No notes or electronic devices (calculators, iPhones, iPads, etc.) are allowed.

Good luck!

[1] Write the TA and time of the section you are going to (but not necessarily registered to).

Those are some very short answers to check your results. They are not representative of what would be a correct written answer in an exam! (except for the last exercice)

1. Each of the five parts below is worth 2 points.

   For this problem only: Please provide short answers (you can add short explanations if you find it appropriate).

   (a) (2 points) If X is a Binomial random variable with parameters 10 and 0.6 what are $\mathbf{E}(X)$ and $\text{var}(X)$?

   Soln: $\mathbf{E}(X) = 10(0.6) = 6$, $\text{var}(X) = 10(0.6)(0.4) = 2.4$

   (b) (2 points) 72 people each toss a fair (6 sided) die twice. Let X be the number of people who get snake eyes (i.e., a 1 on each die). What is $\mathbf{P}(X = 3)$?

   X follows $Bin(72, 1/36)$. So

   (c) (2 points) Write two formulas to compute $\text{cov}(X, Y)$

   Soln:

   $$\text{cov}(X,Y) = \mathbf{E}(XY) - \mathbf{E}(X)\mathbf{E}(Y) = \mathbf{E}((X - \mathbf{E}X)(Y - \mathbf{E}Y))$$

   (d) (2 points) Let X be the number of tosses of a fair coin repeatedly until there are 100 heads in total. Find $\text{var}(X)$.

   Soln: X is the sum of 100 independent geometric random variables with parameter 1/2. So

   $$\text{var}(X) = 100\frac{1 - 1/2}{1/4} = 200$$

(e) (2 points) If X and Y are independent and both have variance 1, compute $\mathrm{var}(2X + 3Y)$.

Soln:

$$\mathrm{var}(2X + 3Y) = 4\,\mathrm{var}(X) + 9\,\mathrm{var}(Y) = 13.$$

2. Let $X_1, X_2, \cdots, X_n$ be independent random variables with Bernoulli distribution of parameter p. Let $A = X_1 + \cdots + X_n$. Fix $k \in \{0, 1, \cdots, n\}$.

   (a) (2 points) Find $\mathbf{E}[X_1 + \cdots X_n | A = k]$.

   (b) (8 points) Find $\mathbf{P}(X_1 = X_2 = 1 | A = k)$. Simplify the answer.

(a) $\mathbf{E}[X_1 + \cdots X_n | A = k] = \mathbf{E}[k | A = k] = k$

(b)

$$\mathbf{P}(X_1 = X_2 = 1 | A = k) = \frac{k(k-1)}{n(n-1)}$$

3. A fair die with 6 faces, numbered 1, 2, 3, 4, 5, 6, is rolled n times. Let X_i be the number of times that the ith face comes up.

   (a) (5 points) Find $\mathbf{E}(X_i)$ and $\text{var}(X_i)$.

   (b) (5 points) Find the PMF of $X_1 + X_2$.

(a) X_i follows $Bin(n, 1/6)$.

(b) $X_1 + X_2$ follows $Bin(n, 1/3)$.

4. Put n balls into r boxes such that each ball has equal probability in any box. Let

$$X_i = \begin{cases} 1 & \text{if there is no ball in the box } i, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (3 points) Find $\mathbf{E}X_i$.

(b) (4 points) Find $\mathbf{E}(X_iX_j)$ and $\text{cov}(X_i, X_j)$ for $i \neq j$.

(c) (3 points) Let S be the number of empty boxes. Find a relation between S and $X_i's$, and then find $\mathbf{E}(S)$.

(a) $\mathbf{P}(X_i = 1) = (\frac{r-1}{r})^n$ and

$$\mathbf{E}(X_i) = \mathbf{P}(X_i = 1) = (\frac{r-1}{r})^n.$$

(b)

$$\mathbf{E}(X_iX_j) = \mathbf{P}(X_i = X_j = 1) = (\frac{r-2}{r})^n$$

$$\text{cov}(X_i, X_j) = \mathbf{E}(X_iX_j) - \mathbf{E}(X_i)\mathbf{E}(X_j) = (\frac{r-2}{r})^n - (\frac{r-1}{r})^{2n}.$$

(c)

$$S = X_1 + \cdots X_r.$$

$$\mathbf{E}(S) = \mathbf{E}(X_1) + \cdots \mathbf{E}(X_n) = r(\frac{r-1}{r})^n$$

5. Let $X_1, X_2, \cdots$ be independent random variables with the $Bin(5, \frac{1}{2})$ distribution, and T be a random variable independent of the $X_i's$ with the geometric distribution with parameter p. Let $S_n = X_1 + X_2 + \cdots + X_n$, and $S_T = X_1 + \cdots X_T$.

   (a) (2 points) What is the distribution of S_n?

   (b) (2 points) What is $\mathbf{E}X_i$?

   (c) (3 points) Find $\mathbf{E}[S_T|T = n]$.

   (d) (3 points) Find $\mathbf{E}S_T$. Simplify your answer.

(a) S_n follows $Bin(5n, \frac{1}{2})$

(b) $\mathbf{E}(X_i) = 5/2$.

(c) $\mathbf{E}[S_T|T = n] = \frac{5n}{2}$.

(d)

$$\mathbf{E}(S_T) = \sum_{n=1}^{\infty} \mathbf{E}(S_T|T = n)\mathbf{P}(T = n) = \sum_{n=1}^{\infty} \frac{5n}{2}\mathbf{P}(T = n) = \frac{5}{2}\mathbf{E}(T) = \frac{5}{2p}$$

6. (10 points) Pair of random variables (X, Y) has range

$$\{(2^{-k}, 2^{-l}) | l, k \text{ are integers and } 1 \leq k < l\}.$$

The joint probability mass function is given by

$$p_{X,Y}(2^{-k}, 2^{-l}) = C2^{-k-l},$$

for some constant C. Find the value of C. Find the marginal probability mass functions for both X and Y and evaluate $\mathbf{E}(X)$ and $\mathbf{E}(Y)$.

Solution: We get C from the condition that the probabilities have to add up to 1. This gives

$$\sum_{k=1}^{\infty}\left(\sum_{l=k+1}^{\infty} C2^{-k-l}\right) = 1.$$

The inner sum $\sum_{l=k+1}^{\infty} C2^{-k-l}$ is the sum of geometric series whose first element is $C2^{-2k-1}$ and the step factor is $1/2$ so

$$\sum_{l=k+1}^{\infty} C2^{-k-l} = \frac{C2^{-2k-1}}{1-1/2} = C4^{-k}$$

Now the condition is $\sum_{k=1}^{\infty} \frac{C}{4^k} = 1$ that is

$$\frac{C/4}{1-1/4} = 1 \;\Rightarrow\; C = 3.$$

The range of X is 2^{-k} for integers $k \geq 1$ and the marginal PMF of X is

$$\mathbf{P}(X = 2^{-k}) = \sum_{l=k+1}^{\infty} \mathbf{P}(X = 2^{-k}, Y = 2^{-l}) = 3\sum_{l=k+1}^{\infty} 2^{-k-l}.$$

This is geometric series whose first element is 2^{-2k-1} and step factor $1/2$ so

$$\mathbf{P}(X = 2^{-k}) = 3\frac{2^{-2k-1}}{1-1/2} = \frac{3}{4^k}.$$

The range of Y is 2^{-l} for $l \geq 2$ and the marginal PMF of Y is

$$\mathbf{P}(Y = 2^{-l}) = \sum_{k=1}^{l-1} \mathbf{P}(X = 2^{-k}, Y = 2^{-l}) = 3\sum_{k=1}^{l-1} 2^{-k-l} = 3 \cdot 2^{-l-1}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{l-2}}\right)$$

The expression in the sum is equal to $\frac{1-1/2^{l-1}}{1-1/2} = 2 - 2^{-l+2}$, so $\mathbf{P}(Y = 2^{-l}) = 3 \cdot 2^{-l} - 3 \cdot 2^{-2l+1}$.

Now the moments are

$$\mathbf{E}(X) = \sum_{k=1}^{\infty} 2^{-k}\mathbf{P}(X = 2^{-k}) = \sum_{k=1}^{\infty} \frac{3}{8^k} = \frac{3/8}{1 - 1/8} = \frac{3}{7},$$

and

$$\mathbf{E}(Y) = \sum_{l=2}^{\infty} 2^{-l}\mathbf{P}(Y = 2^{-l}) = \sum_{l=2}^{\infty} 3 \cdot 4^{-l} - \sum_{l=2}^{\infty} 6 \cdot 8^{-l} = \frac{3/16}{1 - 1/4} - \frac{6/64}{1 - 1/8} = \frac{1}{4} - \frac{3}{28} = \frac{1}{7}.$$