UCLA MATH 170A, WINTER 2018, MIDTERM II

NAME ______________ STUDENT ID # _

This is a ... note examination. No calculators a...
Please sh... l credit will be given to partial answ...
are 5 prob... ts. Each question is worth 20 points.

PROBLEM	1	2	3	4	5	TOTAL
SCORE	20	20	~~8~~ 13	17	20	~~80~~ 90

SJD

1. Let X be a Poisson random variable with parameter 3. What is $\mathbb{E}[X]$ and $\mathrm{var}[X]$?

X is poisson with $\lambda = 3$

$$\hookrightarrow \sum_{k=0}^{\infty} e^{-3} \frac{3^k}{k!} \rightarrow E[X] = \sum_{k=0}^{\infty} k e^{-3} \frac{3^k}{k!} = e^{-3} \sum_{k=0}^{\infty} k \frac{3^k}{k!}$$

$$= e^{-3} \sum_{k=0}^{\infty} \frac{3^k}{(k-1)!} = 3 \cdot e^{-3} \sum_{k=1}^{\infty} \frac{3^{k-1}}{(k-1)!} = 3 \cdot e^{-3} \cdot e^{3}$$

$$= 3 \checkmark$$

$(-1)!$?? No

pull this 3 out to increment k and keep moving

$$\boxed{E[X] = \lambda = 3}$$

$$\mathrm{var}(x) = E((X-\lambda)^2) = E(X^2) - (E(X))^2$$

Using part A above, $\hookrightarrow = e^{-3} \sum_{k=0}^{\infty} k^2 \frac{3^k}{k!} - (3)^2$

$$= e^{-3} \sum_{k=0}^{\infty} \frac{k\, 3^k}{(k-1)!} - 9$$

$$= 3e^{-3} \sum_{k=1}^{\infty} \frac{k \cdot 3^{k-1}}{(k-1)!} - 9$$

$$= 3 \cdot 4 \cdot e^{-3} \cdot e^{3} - 9 = 12 - 9 = 3$$

$$\boxed{\mathrm{Var}[X] = \lambda = 3}$$

2. Let X and Y be independent random variables with zero means, that is $\mathbb{E}[X] = \mathbb{E}[Y] = 0$. Show that

$$\operatorname{var}(X + Y + XY) = \operatorname{var}(X) + \operatorname{var}(Y) + \operatorname{var}(XY).$$

LHS: $var(X+Y+XY) = E[(X+Y+XY)^2] - (E(X+Y+XY))^2$

$(X+Y+XY)(X+Y+XY) = X^2 + XY + X^2Y + XY + Y^2 + XY^2 + X^2Y + XY^2 + X^2Y^2$

$= X^2 + 2XY + 2X^2Y + 2XY^2 + X^2Y^2$

$= E(X^2 + 2XY + 2X^2Y + 2XY^2 + X^2Y^2) - (E(X)^{\to 0} + E(Y)^{\to 0} + E(XY))^2$

$= E(X^2) + E(2XY) + E(X^2Y) + E(2XY^2) + E(X^2Y^2)$

$- E[XY] \cdot E[XY]$

$= E[X^2] + 2E[X]^{\to 0}E(Y)^{\to 0} + 2E(X^2)E(Y)^{\to 0} + 2E[X]^{\to 0}[Y^2] + E[Y^2]$

$+ E[X^2Y^2] - E[XY]^{\to 0} \cdot E[XY]$

$= E[X^2] + E[Y^2] + E[X^2Y^2]$

RHS: $var(X) + var(Y) + var(XY) = E(X^2) - (E[X]^{\to 0})^2 + E(Y^2) - (E(Y)^{\to 0})^2$

$+ E(X^2Y^2) - E(XY)^{\to 0\,2}$

~~same~~ equal!!

$= E[X^2] + E(Y^2) + E[X^2Y^2]$ ✓

Since we can simplify the LHS and RHS to the same expression, $\overset{LHS}{var(X+Y+XY)} = \overset{RHS}{var(X) + var(Y) + var(XY)}$ ✓

necessarily

not independent

3. In a sample space Ω we have n events $A_1, A_2, \ldots, A_n$, each having probability equal to p. If $n = 3$ and $p > 2/3$, show that there is an outcome $\omega \in \Omega$ which is contained in at least three of the above events.

3 events $\to n = 3$ A_1, A_2, A_3 $\omega \in \Omega$

$p > 2/3$ for each of the events

Take the indicator function $X_i = \begin{cases} 1 & \text{if } \omega \in A_i \geq \frac{2}{3} \\ 0 & \text{if } \omega \notin A_i \leq 1/3 \end{cases}$

Let S, a rand variable, $= \sum_{i=1}^{3} X_i$

$\boxed{3/20}$ $\boxed{\frac{13}{20}}$

$E[S] = \sum_{i=1}^{3} E[X_i] = 3 \cdot E[X_i] = 3 \cdot 2/3 \cdot 1 \geq 2$

$\boxed{\text{Since the expected value of } S > 2, \text{ we know that } \omega \in \Omega \text{ will be at least 3 events}}$

$3 \cdot 1 \cdot \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2 + 2 \cdot 3 \cdot \left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) + 3 \cdot \left(\frac{2}{3}\right)^3 + 1 \cdot \left(\frac{1}{3}\right)^3$

$= \frac{6}{27} + \frac{12}{27} + \frac{24}{27} + \frac{1}{27} = 43/27 > 1$

E of bin rar X_i

Analogy $\to$ Box is empty or full for some sample space of putting balls in the box $\to p(\text{ball in box}) = 2/3$, 3 boxes $\to$ Expect there is a ball in a box is 2 $\to$ 2 boxes will balls in them

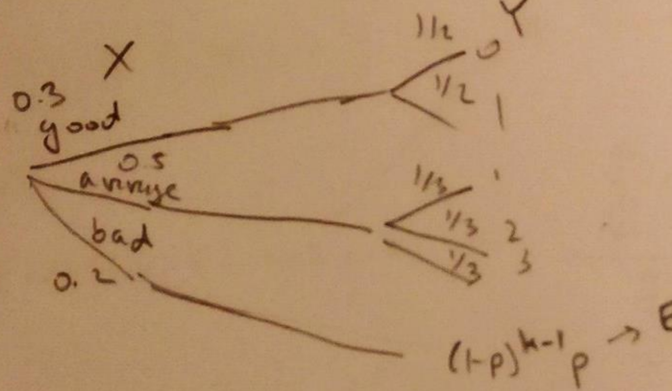
17/20

4. Your favorite sports team plays a good team with probability .3, an average team with probability .5 and a bad team with probability .2.
If they play a good team they score 0 or 1 point with equal probability.
If they play an average team, the number of points they score is uniformly distributed on $\{1,2,3\}$.
If they play a bad team the number the number of points they score is geometrically distributed with parameter 1/2.

$\to E = \frac{1}{p} = \frac{1}{1/2} = 2$

(a) Compute the expected number of points they will score in a game.

(b) If they score 2 points, compute the probability they played an average team.



X = played team

Y = points scored

a) The expected number of points $= \frac{1}{2}\cdot 0.3\,(0) + \frac{1}{2}\cdot 0.3 \cdot 1$ > good ($\frac{3}{10}$)

$+ \frac{1}{3}\cdot\frac{1}{2}\cdot 1 + \frac{1}{3}\cdot\frac{1}{2}\cdot 2 + \frac{1}{3}\cdot\frac{1}{2}\cdot 3$ > average

$+ 0.2 \cdot \frac{1}{1/2}$ > bad

$= 0 + \frac{3}{20} + \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{10} = \frac{3}{20} + 1 + \frac{8}{20} = \boxed{1\frac{11}{20}}$

b) $P(X = \text{avg team} \mid Y=2) = \dfrac{P(X = \text{avg team} \cap Y=2)}{P(Y=2)}$

g = good
a = average
b = bad
total probability

$= \dfrac{P(X=\text{avg team})\cdot P(Y=2 \mid X=\text{avg team})}{P(X=g)\cdot P(Y=2\mid X=g) + P(X=a)\,P(Y=2\mid X=a) + P(X=b)\cdot P(Y=2 \mid X=b)}$

$P(Y=2 \mid X=\text{avg}) = \frac{1/6}{1/2} = \frac{1}{3}$

$= \dfrac{0.5\cdot 1/3}{0 + 0.5\cdot 1/3 + 0.2\cdot 1} = \dfrac{1/6}{1/6 + 2/10} = \dfrac{1/6}{\frac{5}{30}+\frac{6}{30}} = \dfrac{1/6}{11/30} = \boxed{30/66}$

−3

$(\frac{1}{2})^k = \frac{1}{4}$ for $k=2 \to 2$ points

5. The number of injury claims per month is modeled by a random variable N with pmf

$n = 0 \to \infty$

$$p_N(n) = \frac{1}{(n+1)(n+2)}$$

for non-negative integers, n.

Calculate the probability of at least one claim during a particular month, given that there have been at most four claims during that month.

$$P(N \geq 1 \mid N \leq 4) = \frac{P(N \geq 1 \cap N \leq 4)}{P(N \leq 4)}$$

$$= P(N = 1 \text{ or } 2 \text{ or } 3 \text{ or } 4) = \frac{\sum_{k=1}^{4} \frac{1}{(k+1)(k+2)}}{\sum_{k=0}^{4} \frac{1}{(k+1)(k+2)}}$$

$$= \frac{\frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \frac{1}{4\cdot5} + \frac{1}{5\cdot6}}{1/1\cdot2 + 1/2\cdot3 + 1/3\cdot4 + 1/4\cdot5 + 1/5\cdot6} \Rightarrow \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + 1/30 \to \frac{10}{60} + \frac{5}{60} + 3/60 + 2/60$$

$$= \frac{1/3}{\frac{1}{2} + 1/3}$$

$\to \frac{3}{6}\ \frac{2}{6}$

$$= \frac{\frac{1}{3}}{5/6}$$

$$= \boxed{\frac{6}{15}}$$

$\boxed{20}$