UCLA MATH 170A, WINTER 2018, MIDTERM I

N. ______________________ STUDENT ID #

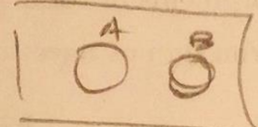
Th te examination. No calculators a
sh t will be given to partial answ
pr question is worth 20 points.

You d carry out the algebraic calculations.

PROBLEM	1	2	3	4	5	TOTAL
SCORE	20	20	18	20	20	98

1. Let A and B be disjoint events such that $\mathbb{P}(A) = 1/4$ and $\mathbb{P}(B) = 1/8$.
(a) Find $\mathbb{P}(A \cap B)$ and $\mathbb{P}(A \cup B^c)$.
(b) Let C and D be independent events such that $\mathbb{P}(C) = 1/2$ and $\mathbb{P}(C \cap D) = 1/3$. Find $\mathbb{P}(D)$, $\mathbb{P}(C \cap D^c)$ and $\mathbb{P}(C|D)$.
(c) On each trial two dice are rolled at same time and the sum of die is recorded. If 20 independent trials are conducted, what is the probability a 3 was recorded exactly 5 times?

$P(A \cap B) = \emptyset$, $P(A) = \frac{1}{4}$, $P(B) = 1/8$



a) $\boxed{P(A \cap B) = 0}$ (disjoint)

$P(A \cup B^c) = 1 - P(B) = 1 - \frac{1}{8} = \boxed{\frac{7}{8} = P(A \cup B^c)}$

b) C and D are independent $P(C) = \frac{1}{2}$, $P(C \cap D) = 1/3$

$\rightarrow P(C \cap D) = P(C) \cdot P(D) \rightsquigarrow P(D) = \frac{1/3}{\frac{1}{2}} = \boxed{\frac{2}{3} = P(D)}$

$P(C \cap D^c) \Rightarrow P(C) \cdot P(D^c) = \frac{1}{2} \cdot \frac{1}{3} = \boxed{1/6 = P(C \cap D^c)}$

$P(C|D) = \frac{P(C \cap D)}{P(D)} = \frac{1/3}{2/3} = \boxed{\frac{1}{2} = P(C|D)}$

c) 20 independent trials, $3^{\leftarrow \text{sum}}$ was recorded exactly 5 times =

$\binom{20}{5} = 5$ rolls w/ 3, $P\overset{\text{sum}}{(3)} = \begin{matrix} 1,2 \\ 2,1 \end{matrix} = \frac{2}{36} = \frac{1}{18}$

$\rightarrow \boxed{\binom{20}{5} \left(\frac{1}{18}\right)^5 \left(\frac{17}{18}\right)^{15}}$

2.(a) A person flips a biased coin which gives a head with probability p. Find the probability of having the third head on the seventh flip;

(b) A box contains $2n$ red and $2n$ blue toys. We select uniformly at random $2n$ toys from the box. Compute the probability that we selected equally many red and blue toys.

a) $P(H) = p$, $P(\sim H) = 1-p$

probability of 3rd head on 7th $\Rightarrow$ in first 6, we have 2 heads and 4 tails, and then 7th = 3rd head = p probability

$\rightarrow \left(\binom{6}{2} p^2 (1-p)^4\right) =$ exactly) p(2 heads in first 6)

$$\boxed{p(\text{3rd head on 7th flip}) = p \cdot \left(\binom{6}{2} p^2 (1-p)^4\right)}$$

b) $2n$ red, $2n$ blue, $\rightarrow$ select $2n$ uniformly

$\rightarrow \binom{4n}{2n} =$ # of ways to choose $2n$ toys

$\binom{2n}{n} =$ # of ways to select n red toys out of $2n$ red

" " blue " " blue

$$\rightarrow \boxed{\frac{\binom{2n}{n}\binom{2n}{n}}{\binom{4n}{2n}}}$$

3. Roll a four sided die twice. Define the following events

$$A = \{ \text{ The sum of the rolls is even } \}$$

$$B = \{ \text{ The sum of the rolls is less than 5} \}$$

$$C = \{ \text{ The rolls give the same number} \}$$

(a) Are A and B independent events?
(b) Are B and C independent events?
(c)Are B and C independent events conditioned on A?

1,1 2,1 3,1
1,2 2,2 3,2
1,3 ~~2,3~~ ~~3,3~~
~~1,4~~ ~~2,4~~
not $\leq$ 5

18/20

$4^2 = 16$ total possibilities

$\frac{6}{16} = \frac{3}{8}$

4 sided die!!

a) $p(A) =$ sum is even $\Rightarrow$ 1,1 2,2 3,3 4,4 13 24 31 42 $\rightarrow \frac{8}{16}$ (16 ← total) $\rightarrow \frac{1}{2} = p(A)$

$p(B) =$ sum is less than 5 $\Rightarrow \frac{1,1 \;\; 1,2 \;\; 1,3 \;\; 2,2 \;\; 2,1 \;\; 3,1}{16} = \frac{6}{16} = \frac{3}{8}$

$p(A \wedge B) =$ even & less than 5 $=$ 1,1, 2,2, 1,3, 3,1 $= \frac{4}{16} = \frac{1}{4}$

$p(A) \cdot p(B) = \frac{1}{2} \cdot \frac{3}{8} = \frac{3}{16} \neq \frac{1}{4}$

A and B are not independent

b) $p(C) =$ same # $\rightarrow$ 1,1 2,2 3,3 4,4 $= \frac{4}{16} = \frac{1}{4}$

$p(B \wedge C) =$ <5 & same # $= \frac{2}{16} = \frac{1}{8}$

$p(C) \cdot p(B) = \frac{1}{4} \cdot \frac{3}{8} = \frac{3}{32} \neq \frac{1}{8}$

B and C are not independent.

c) B and C conditioned on A? indep?

$\frac{1}{4} / \frac{1}{2} = \frac{2}{4} = \frac{1}{2}$

$P(B \mid C \wedge A) \stackrel{?}{=} p(B \mid A) \Rightarrow p(B \mid A) = \frac{1/4}{3/8} = \frac{8}{12} = \frac{2}{3}$

$p(B \mid C \wedge A) = \frac{p(B \wedge C \wedge A)}{p(C \wedge A)} = \frac{1/8}{1/4} = \frac{4}{8} = \frac{1}{2}$ $\quad \frac{2}{3} \neq \frac{1}{2}$

$p(B \wedge C \mid A) \stackrel{?}{=} p(B \mid A) \cdot p(C \mid A)$

$\frac{1/8}{1/2} = \frac{1}{4}$ $\quad \frac{2}{3}$ $\quad \frac{p(C \wedge A)}{p(A)} = \frac{1/4}{1/2} = \frac{1}{2}$

$\hookrightarrow \frac{1}{4} \stackrel{?}{=} \frac{2}{3} \cdot \frac{1}{2} \Rightarrow \frac{1}{4} \neq \frac{1}{3}$

~~B and C are not independent conditioned on A~~

4. A health study tracked a group of persons for five years. At the beginning of the study, 20 percent were classified as heavy smokers, 30 percent as light smokers, and 50 percent as nonsmokers. Results of the study showed that light smokers were twice as likely as nonsmokers to die during the five-year study, but only half as likely as heavy smokers. A randomly selected participant from the study died over the five-year period. Calculate the probability that the participant was a heavy smoker.

0.2 heavy, $\to H$ 0.3 light, $\to L$ 0.5 non-smokers $\to N$

20

p(dieing | light smoker) = 2 × p(Dieing | nonsmoker) = ½ p(dieing | Heavy smoker)

p(heavy smoker | death) $\overset{\text{Bayes Rule}}{=}$ $\dfrac{p(\text{heavy sm}) \cdot p(\text{Death} \mid \text{heavy smoker})}{p(\text{Death})}$

(1) $= \dfrac{p(H) \cdot p(D \mid H)}{p(H) \cdot p(D \mid H) + p(L) \cdot p(D \mid L) + p(N) \cdot p(D \mid N)}$ ← total probability law

assign $p(D \mid L) = x$, then $p(D \mid N) = \frac{1}{2}x$, and $p(D \mid H) = 2x$

$\Rightarrow$ Substitute into (1) above:

$$\frac{0.2 \cdot 2x}{0.2 \cdot 2x + 0.3 \cdot x + 0.5 \cdot \frac{1}{2}x} = \frac{0.4x}{0.4x + 0.3x + 0.25x}$$

$$= \frac{0.4}{0.95} \cancel{\frac{x}{x}} = \boxed{\frac{0.4}{0.95}}$$

✓

5. Just as you shut your apartment door behind you, you relize you might have left your keys inside and locked yourself out. Based on past experience, you figure with probability .8, your keys are inside your apartment and with probability .2 they are in either your right or left pocket, with each being equally likely. You reach into your right pocket, but alas, no keys. What is the probability that your keys are in your left pocket?

I — L — R

p(inside) = 0.8, p(left) = 0.1, p(right) = 0.1

p(~inside) = 0.2   p(~left) = 0.9   p(~right) = 0.9

→ reach into right, no key, prob in left pocket?

$\boxed{20}$

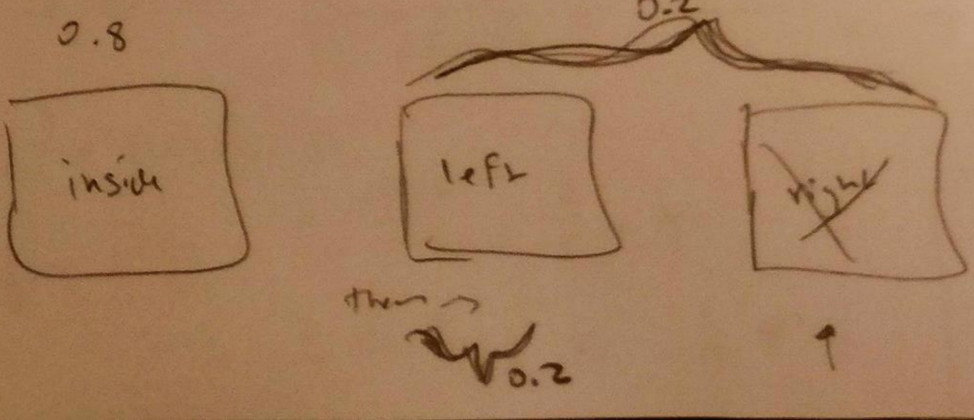
$$p(\text{Left} \mid \sim\text{Right}) = \frac{p(\text{Left} \wedge \sim\text{Right})}{p(\sim\text{Right})} = \frac{0.10}{0.90} \checkmark$$

$$= \boxed{\frac{1}{9}}$$

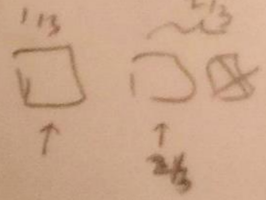
or $p(L \mid \sim R) = \dfrac{p(\sim R \mid L)\cdot p(L)}{p(\sim R)} = \dfrac{1 \cdot 0.1}{0.9}$

conscience check

$= \dfrac{0.1}{0.9} = \dfrac{1}{9}$ ✓



$\dfrac{1/3}{2/3} = \dfrac{1}{2}$ now given the one you just picked not there



$\dfrac{1/3}{2/3} = \dfrac{1}{2}$ which is wrong