(1) (a) Define what it means for events $A_1, A_2, \ldots, A_n$ to be (mutually) *independent.*

Let $S = \{1, 2, \ldots, n\}$

For any subset $s \subseteq S$

$$\mathbb{P}\left(\bigcap_{k \in s} A_k\right) = \prod_{k \in s} \mathbb{P}(A_k) \checkmark$$

So we have $2^n - (n+1)$ subsets to check.

(b) Define what it means for a random variable X to be Poisson(λ) distributed.

$$P_X(k) = \begin{cases} \dfrac{\lambda^k e^{-\lambda}}{k!}, & \text{if } k = 0, 1, 2, 3, \ldots \\ 0, & \text{otherwise} \end{cases} \checkmark$$

4 | 3 | 2 | 1

(c) A lock is manufactured with a $\boxed{\text{four-digit code;}}$ it is generated randomly with each digit chosen independently and uniformly from $\{0, 1, \ldots, 9\}$. What is the probability that the digits in a code add up to 10?

This is the same as put three bars into 13 locations

$$\mathbb{P}(\text{code add up to } 10) = \frac{\binom{13}{3} - 4}{10^4}$$

Can't have
10 0 0 0,
0 10 0 0,
0 0 10 0,
0 0 0 10

$$= \frac{13!}{10!\,3!} - 4 \Big/ 10^4$$

Probability: divide by total # of possibilities.

partition is into A & A^c

(2) (a) State Bayes' rule, that is, the means to compute $\mathbb{P}(A|B)$ from $\mathbb{P}(A)$, $\mathbb{P}(B|A)$, and $\mathbb{P}(B|A^c)$.

Suppose $A_1, A_2, \ldots$ An partition Ω, and B is an event

not same scale NO

(assume these events have non-zero prob.)

$$\mathbb{P}(A_1|B) = \frac{\mathbb{P}(A_1 \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A_1)\cdot\mathbb{P}(A_1)}{\mathbb{P}(B|A_1)\cdot\mathbb{P}(A_1) + \mathbb{P}(B|A_2)\cdot\mathbb{P}(A_2) + \cdots + \mathbb{P}(B|A_n)\cdot\mathbb{P}(A_n)}$$

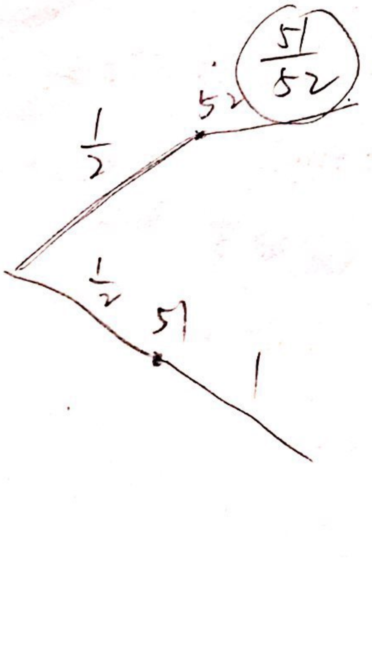
(b) My cupboard has **two** decks of cards, both of which are well shuffled. One deck is a complete set of 52 cards. The other has 51 cards, because the Ace of Spades is missing. Picking a deck uniformly at random, I then observe that the top card is not the Ace of Spades. What is the probability that I am holding the complete deck?

A : the event that I am holding the complete deck

B : the event that the top card is not the Ace of Spades

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\cdot\mathbb{P}(A)}{\mathbb{P}(B|A)\cdot\mathbb{P}(A) + \mathbb{P}(B|A^c)\cdot\mathbb{P}(A^c)}$$

$$= \frac{\frac{51}{52}\cdot\frac{1}{2}}{\frac{51}{52}\cdot\frac{1}{2} + 1\cdot\frac{1}{2}} = \frac{51}{104} \times \frac{104}{103} = \frac{51}{103}$$



$$\frac{51}{104} + \frac{52}{104}$$

$$\frac{103}{104}$$

MA

$12 \quad \overline{11} \quad 6(\frac{1}{6}\cdot\frac{1}{11}) \quad \frac{1}{6}\cdot\frac{10}{11}\cdot\frac{2}{10}$

(3) My dryer contains **SIX pairs** of socks of **different colors**. I remove socks one at a time without looking (the dryer completely randomizes the socks).

(a) What is the probability that the first two socks form a pair?

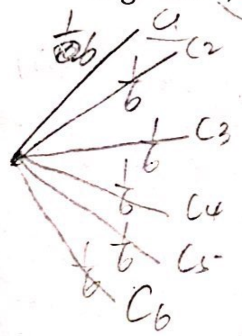
(b) If the first two socks do not match, what is the probability that the third will complete a pair?

(c) What is the probability that a pair can be found among the first three socks? $\quad 12!$

(d) If socks within a pair are declared indistinguishable, in how many different orderings can **all** the socks come out of the dryer.

(e) What is the probability that each member of every pair of socks comes out consecutively? E.g. black, black, brown, brown, red, red, orange, orange, yellow, yellow, green, green.

$\frac{1}{6}\cdot\frac{10}{11}\cdot\frac{2}{10}$



(a) $\mathbb{P} = 6\cdot\left(\frac{1}{6}\cdot\frac{1}{11}\right) = \frac{1}{11}$

(b) $\mathbb{P} = 6\cdot\left(\frac{1}{6}\cdot\frac{10}{11}\cdot\frac{2}{10}\right)$? $= \frac{2}{11}$

$(1-\frac{1}{11})\times$

(c) By (a) & (b), $\mathbb{P} = \frac{1}{11} + \frac{2}{11} = \frac{3}{11}$

(d) $\dfrac{12!}{2!\,2!\,2!\,2!\,2!\,2!} = \dfrac{12!}{2^6}$ $\qquad 5\cdot\left(\frac{1}{5}\cdot\frac{1}{9}\right)$

(e) 12 Socks : $\mathbb{P}$(first two match) $= \frac{1}{11}$

10 Socks : $\mathbb{P}$ ———— $= \frac{1}{9}$

8 socks : $\mathbb{P}$ ———— $= \frac{1}{7}$

6 socks : $\mathbb{P}$ ———— $= \frac{1}{5}$

4 socks : $\mathbb{P}$ ———— $= \frac{1}{3}$

2 socks : $\mathbb{P}$ ———— $= 1$

Final Answer:

$\mathbb{P} = \frac{1}{11}\times\frac{1}{9}\times\frac{1}{7}\times\frac{1}{5}\times\frac{1}{3}\times 1 = \frac{1}{11\times 9\times 7\times 5\times 3}$

(4) Consider the uniform probability law on the sample space $\Omega = \{a, b, c, d, e\}$, as well as two random variables X and Y whose values are as follows:

	a	b	c	d	e
X	1	1	2	2	1
Y	1	2	3	2	-1

1 2 6 4 -1

Determine the following:

(a) $\mathbb{E}(X)$ and $\operatorname{var}(X)$.

(b) $p_{X,Y}(2,2)$.

(c) $\mathbb{E}(Y - X)$ and $\boxed{\mathbb{E}(XY)}$.

(d) The event $X = 1$.

(e) $\mathbb{P}(X = Y | X = 1)$.

(e)
$$\mathbb{P}(X=Y \mid X=1) = \frac{\mathbb{P}(X=Y \cap X=1)}{\mathbb{P}(X=1)} = \frac{\frac{1}{5}}{\frac{3}{5}} = \frac{1}{3} ✓$$

(a)
$$P_X k = \begin{cases} \frac{3}{5}, & \text{if } k=1 \\ \frac{2}{5}, & \text{if } k=2 \end{cases}$$

$$\mathbb{E}(X) = 1 \cdot \frac{3}{5} + 2 \cdot \frac{2}{5} = \frac{7}{5} ✓$$

$$\mathbb{E}(X^2) = 1 \cdot \frac{3}{5} + 4 \cdot \frac{2}{5} = \frac{11}{5}$$

$$\operatorname{Var}(X) = \frac{11}{5} - \left(\frac{7}{5}\right)^2 = \frac{11}{5} - \frac{49}{25} = \frac{6}{25} ✓$$

(b)
$$P_{X,Y}(2,2) = \mathbb{P}(X=2, Y=2) = \frac{1}{5} ✓$$

(c)
$$P_Y l = \begin{cases} \frac{1}{5}, & \text{if } k=-1 \\ \frac{1}{5}, & \text{if } k=1 \\ \frac{1}{5}, & \text{if } k=3 \\ \frac{2}{5}, & \text{if } k=2 \end{cases}$$

$$\mathbb{E}(Y) = -\frac{1}{5} + \frac{1}{5} + \frac{3}{5} + \frac{4}{5} = \frac{7}{5}$$

$$\mathbb{E}(Y-X) = \mathbb{E}(Y) - \mathbb{E}(X) = \frac{7}{5} - \frac{7}{5} = 0 ✓$$

$$\mathbb{E}(XY) = \cancel{1 \cdot \frac{1}{5}} + 2 \cdot \frac{1}{5} + 6 \cdot \frac{1}{5} + 4 \cdot \frac{1}{5} \cancel{- 1 \cdot \frac{1}{5}} = \frac{12}{5} ✓$$

(d) A subset w of Ω

$$w = \{a, b, e\} ✓$$