170A Killip | Midterm | October 31st

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Rules.

- There are **FOUR** problems; ten points per problem.
- There are extra pages after problems 3 and 4. You may also use the backs of pages.
- No calculators, computers, notes, books, crib-sheets,...
- Out of consideration for your class-mates, no chewing, humming, pen-twirling, snoring,... Try to sit still.
- Turn off your cell-phone ringer.

1	2	3	4	Σ
10	10	10	10	40

(1) (a) Define what it means for events $A_1, A_2, \ldots, A_n$ to be (mutually) *independent.*

$A_1, A_2, \cdots A_n$ are mutually independent if

$$\bigcap_{k \in S} A_k = \prod_{k \in S} \mathbb{P}(A_k)$$

for every subset S of $\{A_1, A_2, \cdots A_n\}$

(b) Define what it means for a random variable X to be Poisson(λ) distributed.

a random variable X is poisson (λ) distributed if

$$\mathbb{P}(X=k) = P_X(k) = \begin{cases} \frac{e^{-\lambda}\lambda^k}{k!} & k=0,1,2,3,\cdots \\ 0 & \text{otherwise} \end{cases}$$

(c) A lock is manufactured with a four-digit code; it is generated randomly with each digit chosen independently and uniformly from $\{0, 1, \ldots, 9\}$. What is the probability that the digits in a code add up to 10?

$$\mathbb{P}(\text{digits add up to } 10) = \frac{\binom{10+3}{3}-4}{10^4} = \frac{\binom{13}{3}-4}{10^4}$$

(2) (a) State Bayes' rule, that is, the means to compute $\mathbb{P}(A|B)$ from $\mathbb{P}(A)$, $\mathbb{P}(B|A)$, and $\mathbb{P}(B|A^c)$.

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\cdot\mathbb{P}(A)}{\mathbb{P}(B|A)\cdot\mathbb{P}(A) + \mathbb{P}(B|A^c)\cdot\mathbb{P}(A^c)}$$

$$= \frac{\mathbb{P}(B|A)\cdot\mathbb{P}(A)}{\mathbb{P}(B|A)\cdot\mathbb{P}(A) + \mathbb{P}(B|A^c)\cdot(1-\mathbb{P}(A))} \checkmark$$

(b) My cupboard has two decks of cards, both of which are well shuffled. One deck is a complete set of 52 cards. The other has 51 cards, because the Ace of Spades is missing. Picking a deck uniformly at random, I then observe that the top card is not the Ace of Spades. What is the probability that I am holding the complete deck?

deck A : 52 cards

deck B: 51 cards

event X : holding deck A

event Y: top card is not the Ace of Spades

$$\mathbb{P}(X|Y) = \frac{\mathbb{P}(Y|X)\cdot\mathbb{P}(X)}{\mathbb{P}(Y|X)\cdot\mathbb{P}(X) + \mathbb{P}(Y|X^c)\cdot(1-\mathbb{P}(X))}$$

$$= \frac{\frac{51}{52}\times\frac{1}{2}}{\frac{51}{52}\times\frac{1}{2} + 1\times\frac{1}{2}}$$

$$= \frac{\frac{51}{52}}{\frac{51}{52}+1} = \frac{51}{51+52} = \frac{51}{103} \checkmark$$

(3) My dryer contains **SIX pairs** of socks of **different colors**. I remove socks one at a time without looking (the dryer completely randomizes the socks).

(a) What is the probability that the first two socks form a pair?

(b) If the first two socks do not match, what is the probability that the third will complete a pair?

(c) What is the probability that a pair can be found among the first three socks?

(d) If socks within a pair are declared indistinguishable, in how many different orderings can **all** the socks come out of the dryer.

(e) What is the probability that each member of every pair of socks comes out consecutively? E.g. black, black, brown, brown, red, red, orange, orange, yellow, yellow, green, green.

(a) $\mathbb{P} = 1 \times \frac{1}{12-1} = \frac{1}{11}$ ✓

(b) $\mathbb{P} = \frac{2}{12-2} = \frac{2}{10} = \frac{1}{5}$ ✓

(c) $\mathbb{P} = 1 - \left(1 \times \frac{12-1-1}{12-1} \times \frac{12-2-2}{12-2}\right) = 1 - 1 \times \frac{10}{11} \times \frac{8}{10} = 1 - \frac{8}{11} = \frac{3}{11}$ ✓

(d) $\frac{12!}{2!\,2!\,2!\,2!\,2!\,2!} = \binom{12}{2\ 2\ 2\ 2\ 2\ 2}$ ✓

(e) $\mathbb{P} = 1 \times \frac{1}{11} \times 1 \times \frac{1}{9} \times 1 \times \frac{1}{7} \times 1 \times \frac{1}{5} \times 1 \times \frac{1}{3} \times 1 \times \frac{1}{1} \times 6!$ ✓

$$= \frac{\binom{12}{2\ 2\ 2\ 2\ 2\ 2}}{6!}$$

(4) Consider the uniform probability law on the sample space $\Omega = \{a, b, c, d, e\}$, as well as two random variables X and Y whose values are as follows:

	a	b	c	d	e
X	1	1	2	2	1
Y	1	2	3	2	-1

Determine the following:

(a) $\mathbb{E}(X)$ and $\mathrm{var}(X)$.

(b) $p_{X,Y}(2,2)$.

(c) $\mathbb{E}(Y - X)$ and $\mathbb{E}(XY)$.

(d) The event $X = 1$.

(e) $\mathbb{P}(X = Y | X = 1)$.

(a) $P_X(k) = \begin{cases} \frac{3}{5} & k=1 \\ \frac{2}{5} & k=2 \\ 0 & \text{otherwise} \end{cases}$

$\mathbb{E}(X) = \sum_k k P_X(k) = \frac{3}{5} \times 1 + \frac{2}{5} \times 2 = \frac{7}{5}$

$\mathbb{E}(X^2) = \sum_k k^2 P_X(k) = \frac{3}{5} \times 1 + \frac{2}{5} \times 4 = \frac{11}{5}$

$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{11}{5} - \left(\frac{7}{5}\right)^2 = \frac{55}{25} - \frac{49}{25} = \frac{6}{25}$

(b)

$Y \backslash X$	1	2
-1	$\frac{1}{5}$	0
1	$\frac{1}{5}$	0
2	$\frac{1}{5}$	$\frac{1}{5}$
3	0	$\frac{1}{5}$

$P_{X,Y}(2,2) = \frac{1}{5}$

(c) $\mathbb{E}(Y) = \sum_k k P_Y(k) = 1 \times \frac{1}{5} + 2 \times \frac{2}{5} + 3 \times \frac{1}{5} + (-1) \times \frac{1}{5}$
$= \frac{1}{5} + \frac{4}{5} + \frac{3}{5} - \frac{1}{5} = \frac{7}{5}$

$\mathbb{E}(Y - X) = \mathbb{E}(Y) + \mathbb{E}(-X) = \mathbb{E}(Y) - \mathbb{E}(X) = \frac{7}{5} - \frac{7}{5} = 0$

$\mathbb{E}(XY) = \sum_{k,l} kl P_{XY}(k,l) = \frac{1}{5} \times (-1) + \frac{1}{5} \times 1 + \frac{1}{5} \times 2 + \frac{1}{5} \times 4 + \frac{1}{5} \times 6 = \frac{12}{5}$

(d) $\{a, b, e\}$

(e) $\mathbb{P}(X=Y | X=1) = \dfrac{\mathbb{P}(X=Y \cap X=1)}{\mathbb{P}(X=1)} = \dfrac{\frac{1}{5}}{\frac{3}{5}} = \frac{1}{3}$