Midterm 2, Math 170A - Lec. 1, Fall 2016

Instructor: Pierre-François Rodriguez

Printed name: SOLUTIONS

Signed name: ____________________

Student ID number: ____________________

Instructions:

- Read the following problems very carefully.
- The correct final answer alone is not sufficient for full credit - try to explain your answers as much as you can. This minimizes the chance of losing points for not explaining details, and maximizes the chance of getting partial credit if you fail to solve the problem completely.
- You are only allowed to use items necessary for writing. Any additional resources, in particular **notes, books, sheets, and any electronic devices, are not allowed.** In case you need more paper please raise your hand (you may write on the back pages).

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. Each of the five parts below is worth 2 points. All unspecified random variables appearing in this problem are assumed to be discrete.

(a) (2 points) Let X be a random variable with $\mathbf{E}[X] = 1$ and $\text{var}(X) = 2$. Compute $\mathbf{E}[X^2]$.

$E[X^2] = var(X) + E[X]^2 = 2+1 = 3$

(b) (2 points) Let X be a random variable with $\mathbf{E}[X] = 1$, $\text{var}(X) = 2$, and $\mathbf{E}[X^4] = 4$. Compute $\text{var}(X^2)$.

There is no such RV.

$\rightarrow$ 2 points free

(c) (2 points) Let X be a Poisson random variable with parameter 3, i.e. its PMF is given by $p_X(k) = e^{-3}3^k/k!$, for $k \in \{0, 1, 2, \ldots\}$. Determine $\mathbf{P}(X > 0)$.

$P(X>0) = 1 - P(X=0) = 1 - e^{-3}$

(d) (2 points) A fair coin is flipped repeatedly. Compute the probability that it lands heads for the first time on the 25-th toss.

$\left(\frac{1}{2}\right)^{25}$

(e) (2 points) True or false: there exists a random variable X with $\mathbf{E}[X^{2016}] = -1$.

false : $X^{2016} \geq 0$

$\Longrightarrow E[X^{2016}] \geq 0.$

2. (10 points) Let X have a geometric distribution with parameter $p \in (0,1)$, and $n \geq 1$ be an integer. Define

$$Y_n = \max\{X, n\}.$$

Find the PMF of Y_n.

Y_n takes on values $n, n+1, n+2, \ldots$

$$P_{Y_n}(n) = P(Y_n = n) = P(\max\{X,n\} = n) = P(X \leq n)$$

$$= \sum_{k=1}^{n} (1-p)^{k-1} p = p \sum_{l=0}^{n-1} (1-p)^{l}$$

$$= p \cdot \frac{1-(1-p)^n}{1-(1-p)} = 1-(1-p)^n$$

For $k = n+1, n+2, \ldots$

$$P_{Y_n}(k) = P(\max\{X, n\} = k) \overset{k>n}{=} P(X = k)$$

$$= (1-p)^{k-1} p.$$

So
$$P_{Y_n}(k) = \begin{cases} 1-(1-p)^n, & k = n \\ (1-p)^{k-1} p, & k = n+1, n+2, \ldots \end{cases}$$

3. Let X be an exponential random variable with parameter $\lambda > 0$, i.e.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{else,} \end{cases}$$

and let $A = \{X \geq 1\}$.

(a) (2 points) Compute $\mathbf{P}(A)$.

$$\mathbb{P}(A) = \int_1^\infty f_X(x)\,dx = -e^{-\lambda x}\Big|_{x=1}^{x=\infty} = e^{-\lambda}$$

(b) (3 points) Find $f_{X|A}(x)$.

$$f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{P(A)} = \lambda e^{-\lambda(x-1)} & , x \geq 1 \\ 0 & , \text{else} \end{cases}$$

(c) (5 points) Compute $\mathbf{E}[X|A]$.

$$E[X|A] = \int_{-\infty}^{\infty} x f_{X|A}(x)\,dx = \int_1^\infty x\lambda e^{-\lambda(x-1)}\,dx$$

$$= e^{\lambda}\left[x\cdot(-e^{-\lambda x})\Big|_{x=1}^{\infty} + \int_1^\infty e^{-\lambda x}\,dx \right]$$

$$= e^{\lambda}\left[+e^{-\lambda} + \frac{(-e^{-\lambda x})}{\lambda}\Big|_1^\infty \right] = e^{\lambda}\left[+e^{-\lambda} + \frac{e^{-\lambda}}{\lambda} \right]$$

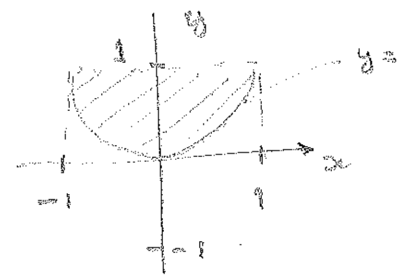
$$= 1 + \frac{1}{\lambda}$$

4. A point (X, Y) is chosen uniformly on the unit square $[-1,1]^2 = \{(x,y) \in \mathbb{R}^2 | -1 \le x \le 1, -1 \le y \le 1\}$.

(a) (2 points) Write down the joint PDF $f_{X,Y}(x,y)$.

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}, & (x,y) \in [-1,1]^2 \\ 0, & \text{else} \end{cases}$$

(b) (8 points) Compute $\mathbf{P}(X^2 < Y)$.



$$P(X^2 < Y)$$

$$= \int_{\{(x,y) \mid y > x^2\}} dx \int dy \; f_{X,Y}(x,y)$$

$$= \int_{-1}^{1} dx \int_{x^2}^{1} dy \; \frac{1}{4} = \int_{-1}^{1} dx \; \frac{1}{4}(1-x^2) = \frac{1}{4}\left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1}$$

$$= \frac{1}{4}\left(2 - \frac{2}{3}\right) = \frac{1}{3}$$

5. Let $n, r \geq 3$ be integers. Place n balls independently into r boxes such that each ball has equal probability to be in any box. For $1 \leq i \leq n$, let

$$X_i = \begin{cases} 1 & \text{if there is no ball in the box } i, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (2 points) Find $\mathbf{E}[X_i]$.

$$E[X_i] = P(X_i = 1) = \left(1 - \frac{1}{r}\right)^n$$

(b) (2 points) Let S be the number of empty boxes. Express S in terms of the X_i's and find $\mathbf{E}[S]$.

$$S = \sum_{i=1}^{r} X_i$$

$$E[S] = \sum_{i=1}^{r} E[X_i] \overset{(a)}{=} r \cdot \left(1 - \frac{1}{r}\right)^n$$

(c) (6 points) Compute $\mathbf{E}[X_1 X_2]$. *Hint:* this is a probability.

$$E[X_1 X_2] = P(X_1 = 1, X_2 = 1)$$

$$= P(X_1 = 1) \cdot P(X_2 = 1 \mid X_1 = 1)$$

$$\overset{(a)}{=} \left(1 - \frac{1}{r}\right)^n \cdot \left(1 - \frac{1}{r-1}\right)^n$$

$$= \left(\frac{r-2}{r}\right)^n$$