Midterm 1, Math 170A - Lec. 1, Fall 2016

Instructor: Pierre-François Rodriguez

Printed name: SOLUTIONS

Signed name: ______________________

Student ID number: ______________________

Instructions:

- Read the following problems very carefully.
- The correct final answer alone is not sufficient for full credit - try to explain your answers as much as you can. This minimizes the chance of losing points for not explaining details, and maximizes the chance of getting partial credit if you fail to solve the problem completely.
- You are only allowed to use items necessary for writing. Any additional resources, in particular **notes, books, sheets, and any electronic devices, are not allowed**. In case you need more paper please raise your hand (you may write on the back pages).

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. Each of the five parts below is worth 2 points. A six-sided die is tossed twice. All outcomes are equally likely. Let S be the sum of the two outcomes, $A = \{S \text{ is even}\}$ and $B = \{S \leq 4\}$.

(a) (2 points) Compute $P(A)$.

$$P(A) = \frac{6 \cdot 3}{36} = \frac{1}{2}$$

(for each of 6 possible outcomes for the 1st roll, 3 outcomes for the second yield S even)

(b) (2 points) Compute $P(B)$.

$$P(B) = \frac{3+2+1}{36} = \frac{1}{6}$$

(c) (2 points) Find $P(A|B)$.

$$P(A \cap B) = P(11) + P(13) + P(22) + P(31) = \frac{1}{9}$$

$$\Rightarrow P(A|B) = \frac{2}{3}$$

(d) (2 points) Are A and B independent?

no. $P(A) \neq P(A|B)$

(e) (2 points) Let Ω be an appropriate sample space for this experiment. Are Ω and A independent?

Yes: $P(\Omega \cap A) = P(A) = \underbrace{P(\Omega)}_{=1} \cdot P(A)$;

2. (10 points) A population consists of 40 individuals of type I, 60 individuals of type II, and 100 individuals of type III. A sample of size 30 is taken without replacement.

   a) Find P(the sample contains 10 individuals of type I).

$$= \frac{\binom{40}{10}\binom{160}{20}}{\binom{200}{30}}$$

   b) Find P(the sample contains 10 individuals of type I and 10 individuals of type II).

$$= \frac{\binom{40}{10}\binom{60}{10}\binom{100}{10}}{\binom{200}{30}}$$

3. (10 points) A box contains 4 red and 4 black balls. A ball is drawn randomly from the box. If it is black, it is simply returned to the box. If it is red, it and 4 more additional red balls are returned to the box. Then a second ball is drawn from the box. Define the events

$$B_1 = \{\text{the first ball is black}\}, \qquad B_2 = \{\text{the second ball is black}\}.$$

a) Find $P(B_2)$.

Apply law of total probability:

$$P(B_2) = P(B_2 | B_1) P(B_1) + P(B_2 | B_1^c) P(B_1^c)$$

$$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2}$$

$$= \frac{5}{12}$$

b) Find $P(B_1 \,|\, B_2^c)$.

Bayes' rule:

$$P(B_1 | B_2^c) = P(B_2^c | B_1) \cdot \frac{P(B_1)}{P(B_2^c)}$$

$$= \frac{1}{2} \cdot \frac{\frac{1}{2}}{\frac{7}{12}} = \frac{3}{7}.$$

4. (10 points) Let A, B, C be three events on some probability space (Ω, P), and assume that $P(C) > 0$.

   a) Write down the definition of A, B and C being independent.

$$(1) \quad P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

$$(2) \quad \begin{cases} P(A \cap B) = P(A) P(B) \\ P(B \cap C) = P(B) P(C) \\ P(A \cap C) = P(A) P(C) \end{cases}$$

   c) Recall that A and B are conditionally independent given C if

$$P(A \cap B|C) = P(A|C) \cdot P(B|C).$$

   Show that, if A, B and C are independent, then A and B are conditionally independent given C.

$$P(A \cap B|C) = \frac{P(A \cap B \cap C)}{P(C)} \stackrel{(1)}{=} P(A) P(B)$$

$$P(A|C) P(B|C) = \frac{P(A \cap C)}{P(C)} \frac{P(B \cap C)}{P(C)} \stackrel{(2)}{=} P(A) P(B).$$

$$\Longrightarrow \quad P(A \cap B|C) = P(A|C) P(B|C).$$

5. (10 points) A burglar stole a keychain which has n keys, exactly two of which open a certain apartment door. He tries to open the door using the keys one by one in a random order. Whenever a key does not open the door, he does not attempt to use this key again. Compute the probability that the burglar opens the door on the r-th attempt, for $1 \leq r < n$.
Hint: use the events $A_k = \{$the burglar fails to open the door on the k-th attempt$\}$, for $1 \leq k < n$.

Let $B_r = \{$the burglar opens the door on the r-th attempt$\}$.

want: $P(B_r)$ Note:

$$B_r = A_1 \cap A_2 \cap \dots \cap A_{r-1} \cap A_r^c = \bigcap_{k=1}^{r-1} A_k \cap A_r^c$$

So $P(B_1) = P(A_1^c) = \frac{2}{n}$ and for $r \geq 2$.

$$P(B_r) = P\left(\bigcap_{k=1}^{r-1} A_k \cap A_r^c\right)$$

mult rule

$$= P(A_1)P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \cdots P(A_{r-1}|A_1 \cap \dots \cap A_{r-2})$$

$$\times P(A_r^c|A_1 \cap \dots \cap A_{r-1})$$

$$= \frac{n-2}{n} \frac{n-3}{n-1} \cdots \frac{n-r}{n-r+2} \frac{2}{n-r+1}$$

$$= \frac{2(n-r)}{n(n-1)}$$

using $P(A_k | \underbrace{A_1 \cap \dots \cap A_{k-1}}_{\text{failed in the first } k-1 \text{ attempts}}) = \frac{n-k-1}{n-k+1}$ for $k \geq 1$