1 Five cards are dealt from a 52 card deck. What is the probability of a "two pair" hand? (two pair is two of one rank, two of another rank, and one of a third rank)

$\binom{52}{5}$ = total 5-card hands

$$\underbrace{13\cdot 12\cdot 11}_{\text{choose ranks}}\cdot\underbrace{\binom{4}{2}\binom{4}{2}\binom{4}{1}}_{\text{choose 2 of 1, 2 of 2nd, 1 of 3rd}}$$

$$p(\text{two pair hand}) = \frac{13\cdot 12\cdot 11\cdot \frac{4!}{2!2!}\cdot\frac{4!}{2!2!}\cdot\frac{4!}{1!3!}}{\frac{52!}{5!47!}}$$

$$= \frac{13\cdot 12\cdot 11\cdot \frac{4\cdot 3\cdot \cancel{2}\cdot 4\cdot 3\cdot \cancel{2}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}}{\cancel{2}\cdot \cancel{2}\cdot \cancel{2}\cdot \cancel{2}\cdot \cancel{3}\cdot \cancel{2}}}{\frac{52!}{5!47!}}$$

$$= \frac{13\cdot 12\cdot 11\cdot 12\cdot 12\cdot 5!\,47!}{52!}$$

2 A test has normally distributed scores with mean 100 and 93.3th percentile score 122.5.

(a) What is the standard deviation of the distribution of scores?

(b) What is the probability of a randomly chosen score being between 90 and 110?

$\mu = 100$

93.3^{th} percentile $\rightarrow 1.5\ \sigma$

a) $100 + 1.5\sigma = 122.5$

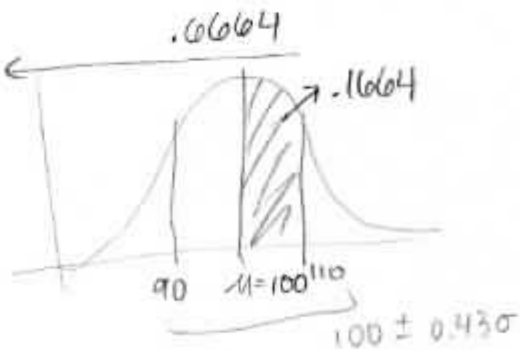
$1.5\sigma = 22.5$

$\sigma = \frac{22.5}{1.5}$

std. dev $\boxed{\sigma = 15}$

$$\begin{array}{r} 1.5 \\ \times 15 \\ \hline 75 \\ 150 \\ \hline 22.5 \end{array}$$

b)



$100 + 10 \rightarrow 0.43\sigma$ above

$10 = 15x$

$\frac{10}{15} = \frac{2}{3} = 0.\overline{66} \rightarrow 0.43\ \sigma$

$$\begin{array}{r} .6664 \\ -\ .5000 \\ \hline .1664 \\ \times\quad 2 \\ \hline .3328 \end{array}$$

probability that random score is between 90 + 100 = $\boxed{.3328}$

3 Explain why

$$\int_1^n \ln x\, dx \qquad -(\ln 2 + \ln + \ldots + \ln(n-1) + (1/2)(\ln n))$$

has a finite, positive limit as n goes to infinity.
[You may assume the |f|, |f"| item but state it clearly if you use it]

$\ln n! = \ln 1 + \ln 2 + \ldots + \ln n$

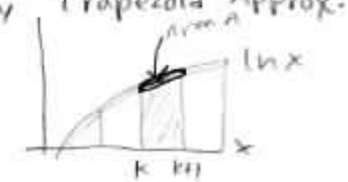
From $\max|f| \le \max|f''|$

with $f = \ln x - L$, L = top of trapezoid

the Area A (in picture to right) $\le |(\ln x)''|\Big|_{k-1}^{k} = \frac{1}{x^2}\Big|_{k-1}^{k} = \frac{1}{k^2}$

so $\int_k^{k+1} \ln x \le \frac{1}{2}\ln k + \frac{1}{2}\ln(k+1) + \frac{1}{k^2}$ (since A is at most $1/k^2$)

Trapezoid Approx:



trapezoid area = $\frac{1}{2}\ln k + \frac{1}{2}\ln k+1$

$\int_k^{k+1} \ln x = \frac{1}{2}\ln k + \frac{1}{2}\ln(k+1) + \text{Area } A \text{ (top)}$

So $E_n = \int_1^n \ln x\, dx - (\frac{1}{2}\ln 1 + \ldots \ln(n-1) + \frac{1}{2}\ln n)$

$= \int_1^n \ln x\, dx - \left(\ln n! - \frac{1}{2}\ln n\right)$

$e^{(E_n)} = e^{(n\ln(n) - n + 1 - \ln n! + \frac{1}{2}\ln n)}$

$= n^n e^{-n} e \cdot (n!)^{-1} \sqrt{n}$

$= \dfrac{\left(\frac{n}{e}\right)^n \sqrt{n} \cdot e}{n!}$

$\lim_{n\to\infty} \dfrac{\left(\frac{n}{e}\right)^n \sqrt{n}}{n!} = \frac{1}{e} \cdot e^{\lim E_n}$ = positive constant

4 Let X and Y be random variables on a sample space which is a finite set

(a) Explain why   E(X) +E(Y)= E(X+Y)

(b) Define what it means for X and Y to be independent

(c) Explain why

variance(X+Y) = variance(X) + variance (Y)

if X and Y are independent . You may assume for this part that E(X)=E(Y)=0 for simplicity in the calculation.

Suggestion for parts (a) and (c): Partition the sample space into sets A(a,b)= set of points x where X(x)=a and Y(x) =b.

a) $E(X)+E(Y)=E(X+Y)$ (X, Y do not have to be independent)

Definition of $E[X]$: $E[X]=\sum_x x\, p_x(k) \rightarrow E(X+Y)=\sum_{x,y}(x+y)p(k)$

(where x = value of X, y = value of Y) $= E(X)+E(Y)=\sum_x x\,p(k)+\sum_y y\,p(k)=\sum (x+y)\,p(k)$

b) If X and Y are independent,

$E[XY]=E[X]E[Y]$; the value of one random variable does not affect the value of the other, so $E[XY]=E[X]E[Y]$

c) Sets $A(a,b)$

$X(x)=a,\ Y(x)=b$

$E[X]=E[Y]=0 \rightarrow E[X+Y]=E[X]+E[Y]=0$

$var(X)=E(X^2)-E(X)^2$

$var(Y)=E(Y^2)-E(Y)^2$

$var(X)+var(Y)=E(X^2)+E(Y^2)-E(X)^2-E(Y)^2$

$=E[X^2]+E[Y^2]+E[X]E[Y]+E[X]E[Y]$

$variance(X)=E(X^2)-E(X)^2=E[(X-E[X])^2]$

$var(X+Y)=E[((x+y)-0)^2]=E[(x+y)^2]$

$=E[x^2+2xy+y^2]=E[x^2]+E[2xy]+E[y^2]$

$=E[x^2]+E[\cancel{y}^2]+2E[XY]=$

when X, Y independent, $E[XY]=E[X]E[Y] \longrightarrow E[X^2]+E[Y^2]+E[X]E[Y]+E[X]E[Y]$

$=var(X)+var(Y)$

5 You have two coins. One is a regular fair coin. One has heads on both sides. You pick a coin at random and throw it five times. Heads comes up all five times. What is the probability that you chose the heads only coin. Explain your answer! .

You either got the double headed coin, or you got the fair coin and got 5 heads.

$P(DH) + P(\text{fair} \,\&\, 5\,H) \Rightarrow$ total possibilities.

double headed

$P(DH) = \frac{1}{2}$ $\quad P(\text{fair} \,\&\, 5\,H) = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)^6$

$$p(DH \mid \text{you get } 5\,H) = \frac{P(DH)}{P(DH) + P(\text{fair} \,\&\, 5\,H)} = \frac{\frac{1}{2}}{\frac{1}{2} + \left(\frac{1}{2}\right)^6}$$

(This is: $\frac{\text{prob. that you got DH coin}}{\text{total prob. of rolling 5 Heads}}$)

$$= \frac{\frac{1}{2}}{\frac{32}{64} + \frac{1}{64}} = \frac{1/2}{33/64}$$

$$= \frac{1}{2} \frac{64}{33} = \frac{64}{66} = \boxed{\frac{32}{33}}$$

$\frac{32}{33}$ chance you chose heads only coin.

y	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

The standard normal table. The entries in this table provide the numerical values of $\Phi(y) = \mathbf{P}(Y \le y)$, where Y is a standard normal random variable, for y between 0 and 3.49. For example, to find $\Phi(1.71)$, we look at the row corresponding to 1.7 and the column corresponding to 0.01, so that $\Phi(1.71) = .9564$. When y is negative, the value of $\Phi(y)$ can be found using the formula $\Phi(y) = 1 - \Phi(-y)$.