Second midterm solution

1(i).

$$\begin{aligned}\mathbb{E}(X+1)I(X\geq 2) &= \mathbb{E}(X+1)-\mathbb{E}(X+1)I(X=0)-\mathbb{E}(X+1)I(X=1)\\ &= (\mathbb{E}X+1)-\mathbb{E}I(X=0)-\mathbb{E}2I(X=1)\\ &= \mathbb{E}X+1-\mathbb{P}(X=0)-2\mathbb{P}(X=1)\\ &= 0.9+1-e^{-0.9}-2e^{-0.9}0.9=1.9-2.8e^{-0.9}.\end{aligned}$$

1(ii).

$$\begin{aligned}\mathrm{Var}(XY) &= \mathbb{E}((XY)^2)-(\mathbb{E}(XY))^2\\ &= \mathbb{E}(X^2Y^2)-(\mathbb{E}X)^2(\mathbb{E}Y)^2\\ &= (\mathbb{E}X^2)(\mathbb{E}Y^2)-(\mathbb{E}X)^2(\mathbb{E}Y)^2.\end{aligned}$$

$$\mathbb{E}X=2 \text{ and } \mathrm{Var}(X)=2$$

so

$$\mathbb{E}(X^2)=\mathrm{Var}(X)+(\mathbb{E}X)^2=6.$$

$$\mathbb{E}Y=8\cdot 0.5=4 \text{ and } \mathrm{Var}(Y)=8\cdot 0.5\cdot(1-0.5)=2$$

so

$$\mathbb{E}(Y^2)=\mathrm{Var}(Y)+(\mathbb{E}Y)^2=18.$$

Therefore,

$$\mathrm{Var}(XY)=6\cdot 18-2^2\cdot 4^2=108-64=44.$$

1(iii).

$$\begin{aligned}\mathbb{E}2^X &= \sum_{k=1}^{\infty}2^k\mathbb{P}(X=k)\\ &= \sum_{k=1}^{\infty}2^k(1-0.9)^{k-1}0.9\\ &= 0.9\sum_{k=1}^{\infty}2^k\cdot(0.1)^{k-1}\\ &= 0.9\cdot 2\sum_{k=1}^{\infty}(0.2)^{k-1}\\ &= 1.8\sum_{k=0}^{\infty}(0.2)^k=\frac{1.8}{1-0.2}=\frac{9}{4}.\end{aligned}$$

2. For $n \in \{1,3,4,5,6\}$, the probability of getting n is

$$\frac{1}{6}\lambda + 0 \cdot (1-\lambda) = \frac{\lambda}{6}.$$

The probability of getting 2 is

$$\frac{1}{6}\lambda + 1 \cdot (1-\lambda) = 1 - \frac{5}{6}\lambda.$$

The probability of getting 2,1,2,2,3,4,2,5 is

$$\left(\frac{\lambda}{6}\right)^4 \left(1 - \frac{5}{6}\lambda\right)^4.$$

Maximizing this is equivalent to maximizing $\frac{\lambda}{6}\left(1 - \frac{5}{6}\lambda\right)$. Taking derivative, we see that the maximum is attained when $\lambda = \frac{3}{5}$.

3. The number of defective items is $X \sim \text{binom}(500, \frac{\lambda}{100}) \approx \text{Poisson}(5\lambda)$.

$$\mathbb{P}(\lambda = 1 | X = 4) = \frac{\mathbb{P}(X = 4 | \lambda = 1)\mathbb{P}(\lambda = 1)}{\mathbb{P}(X = 4)} = \frac{e^{-5}\frac{5^4}{4!}\frac{2}{5}}{\mathbb{P}(X = 4)}.$$

$$\begin{aligned} \mathbb{P}(X = 4) &= \mathbb{P}(X = 4 | \lambda = \frac{1}{2})\mathbb{P}(\lambda = \frac{1}{2}) + \mathbb{P}(X = 4 | \lambda = 1)\mathbb{P}(\lambda = 1) \\ &= e^{-2.5}\frac{(2.5)^4}{4!}\frac{3}{5} + e^{-5}\frac{5^4}{4!}\frac{2}{5}. \end{aligned}$$

Therefore,

$$\mathbb{P}(\lambda = 1 | X = 4) = \frac{e^{-5}\frac{5^4}{4!}\frac{2}{5}}{e^{-2.5}\frac{(2.5)^4}{4!}\frac{3}{5} + e^{-5}\frac{5^4}{4!}\frac{2}{5}}.$$

4(i).

$$\mathbb{E}(X - Y)^2 = \text{Var}(X - Y) + (\mathbb{E}(X - Y))^2 = \text{Var}(X) + \text{Var}(Y) + 0 = 2\text{Var}(X).$$

4(ii). Since X and Y are independent and have the same distribution, $f(X)$ and $f(Y)$ are independent and have the same distribution. So by 4(i),

$$2\text{Var}(f(X)) = \mathbb{E}(f(X) - f(Y))^2 \leq \mathbb{E}(X - Y)^2 = 2\text{Var}(X).$$