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Math 170A Second midterm 9 November

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Total score: 30 points Time: 50 minutes

Full name: Taoran Liu (1 point)

1. (6 points) For the following questions, no justification needed. Only the final answer worths credit.

   (i) If $X \sim$ Poisson(0.9), what is $\mathbb{E}(X+1)I(X \geq 2)$? (2 points)

   (ii) If $X \sim$ Poisson(2) and $Y \sim$ binom(8, 0.5) are independent, what is Var(XY)? (2 points)

   (iii) If $X \sim$ Geom(0.9), what is $\mathbb{E}2^X$? (2 points)

i) $X \sim$ Poisson (0.9)

$$\mathbb{E}(X+1)I(X\geq 2) = \mathbb{E}(X+1) - \mathbb{E}(X+1)I(X=0) - \mathbb{E}(X+1)I(X=1)$$

$$= 1.9 - \mathbb{E}I(X=0) - \mathbb{E}2I(X=1)$$

$$= 1.9 - P(X=0) - 2P(X=1)$$

$$= 1.9 - \frac{e^{-0.9}0.9^0}{0!} - 2\left(\frac{e^{-0.9}0.9}{1!}\right)$$

$$= 1.9 - e^{-0.9} - 1.8e^{-0.9}$$

$$= 1.9 - 2.8e^{-0.9}$$

ii) $$\mathrm{Var}(XY) = \mathbb{E}(XY)^2 - (\mathbb{E}(XY))^2$$

$$= \mathbb{E}X^2\mathbb{E}Y^2 - (\mathbb{E}X\,\mathbb{E}Y)^2$$

$$= 2^2 \cdot 4^2 - (2\cdot 4)^2$$

$$= 64 - 64$$

$$= 0\ ?$$

iii) $X \sim$ Geom (0.9)

$$\mathbb{E}2^X = \sum_{n=1}^{\infty} 2^n (P(X=n))$$

$$= \sum_{n=1}^{\infty} 2^n 0.9(0.1)^{n-1}$$

$$= \frac{0.9}{0.1}\sum_{n=1}^{\infty} 2^n (0.1)^n$$

$$= 9\sum_{n=1}^{\infty} 0.2^n$$

$$= 9\left(\frac{0.2}{1-0.2}\right)$$

$$= 9 \cdot \frac{1}{4} = 2.25$$

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2.25

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2. (7 points) A box contains many fair six sided dices and fake dices where all sides are 2. The proportion of fair dices is $0 \le \lambda \le 1$. Eight dices are drawn from the box with replacement and each of them is rolled once. The result is 2,1,2,2,3,4,2,5. What value of λ maximizes the probability of this result? $\mathbb{P}(x=2) = \lambda\left(\frac{1}{6}\right) + (1-\lambda)$

$$= 1 - \frac{5}{6}\lambda$$

$$\mathbb{P}(x=1) = \lambda \frac{1}{6} = \mathbb{P}(x=3) = \mathbb{P}(x=4) = \mathbb{P}(x=5)$$

$$\mathbb{P}(\text{result}) = \left(1-\frac{5}{6}\lambda\right)^4\left(\frac{1}{6}\lambda\right)^4$$

$$= \left(\left(1-\frac{5}{6}\lambda\right)\left(\frac{1}{6}\lambda\right)\right)^4$$

$$= \left(\frac{1}{6}\lambda - \frac{5}{36}\lambda^2\right)^4$$

$$\frac{d\mathbb{P}(\text{result})}{d\lambda} = 4\left(\frac{1}{6}\lambda - \frac{5}{36}\lambda^2\right)^3\left(\frac{1}{6} - \frac{5}{18}\lambda\right) = 0$$

$$\lambda = 0, \frac{6}{5}, \frac{3}{5}$$

$$\frac{d^2\mathbb{P}(\text{result})}{d\lambda^2} = \underbrace{12\left(\frac{1}{6}\lambda - \frac{5}{36}\lambda^2\right)^2\left(\frac{1}{6} - \frac{5}{18}\lambda\right)^2}_{\text{always +ve}} + 4\left(\frac{1}{6}\lambda - \frac{5}{36}\lambda^2\right)^3\left(\frac{-5}{18}\right)$$

$0 \to$ $\frac{1}{6}\left(\frac{3}{5}\right) - \frac{5}{36}\left(\frac{9}{25}\right) = \frac{1}{10} - \frac{1}{20} = \frac{1}{20}$

$\frac{6}{5} \to$

$$4\left(\frac{1}{20}\right)^3\left(\frac{-5}{18}\right) \Rightarrow -\text{ve}, \quad \text{CCD}.$$

$\boxed{\lambda = \frac{3}{5}}$ is a maximum

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3. (7 points) A factory just bought a machine. Items produced by this machine have a defective rate of $\lambda/100$. Based on information from the machine supplier,

$$\mathbb{P}\left(\lambda = \frac{1}{2}\right) = \frac{3}{5} \text{ and } \mathbb{P}(\lambda = 1) = \frac{2}{5}.$$

To find out this λ, we take 500 sample items and 4 of them are found to be defective. Estimate the probability that $\lambda = 1$.

$X \sim \text{Binom}(500, \lambda/100)$

$\approx X \sim \text{Poisson}(5\lambda)$

$$\mathbb{P}(\underbrace{\lambda=1}_{A} \mid \underbrace{4 \text{ defective}}_{B}) = \frac{\mathbb{P}(B \mid A)\,\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{\frac{e^{-5}5^4}{4!}\left(\frac{2}{5}\right)}{\mathbb{P}(B)}$$

$$\mathbb{P}(B) = \mathbb{P}(X=4 \mid \lambda=1)\mathbb{P}(\lambda=1) + \mathbb{P}(X=4 \mid \lambda=\tfrac{1}{2})\mathbb{P}(\lambda=\tfrac{1}{2})$$
$$= \frac{e^{-5}5^4}{4!}\left(\frac{2}{5}\right) + \left(\frac{3}{5}\right)\left(\frac{e^{-2.5}2.5^4}{4!}\right)$$

$$\mathbb{P}(\lambda=1 \mid 4 \text{ def}) = \frac{\frac{e^{-5}5^4}{4!}\left(\frac{2}{5}\right)}{\frac{e^{-5}5^4}{4!}\cdot\frac{2}{5} + \frac{e^{-2.5}2.5^4}{4!}\cdot\frac{3}{5}} \quad \left(\frac{4!}{4!}\right)\left(\frac{5}{5}\right)$$

$$= \frac{2e^{-5}5^4}{2e^{-5}5^4 + 3e^{-2.5}2.5^4} \cdot \frac{e^5}{e^5}$$

$$= \frac{2\cdot 5^4}{2\cdot 5^4 + 3e^{2.5}2.5^4}$$

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4. (9 points) Suppose that X and Y are independent discrete random variables and have the same distribution.

(i) Show that $\mathbb{E}(X-Y)^2 = 2\mathrm{Var}(X)$. (4 points)

(ii) Show that if $f : \mathbb{R} \to \mathbb{R}$ is 1-Lipschitz, i.e.,

$$|f(x) - f(y)| \leq |x - y| \text{ for all } x, y \in \mathbb{R},$$

then $\mathrm{Var}(f(X)) \leq \mathrm{Var}(X)$. (5 points)

i) $\mathbb{E}(X-Y)^2 = \mathbb{E}X^2 - 2\mathbb{E}XY + \mathbb{E}Y^2$

$= \mathbb{E}X^2 + \mathbb{E}Y^2 - 2\mathbb{E}X\mathbb{E}Y$

but since X & Y have the same distribution,

$\mathbb{E}X = \mathbb{E}Y$.

$\mathbb{E}(X-Y)^2 = \mathbb{E}X^2 + \mathbb{E}X^2 - 2\mathbb{E}X\mathbb{E}X$

$= 2\mathbb{E}X^2 - 2(\mathbb{E}X)^2$

$= 2(\mathbb{E}X^2 - (\mathbb{E}X)^2)$

$= 2\,\mathrm{Var}(X)$

ii) $|f(x) - f(y)| \leq |x-y|$, $x, y \in \mathbb{R}$

$(|f(x) - f(y)|)^2 \leq (|x-y|)^2$

Let X, Y be independent random variables.

$\mathbb{E}(f(X) - f(Y))^2 \leq \mathbb{E}(X-Y)^2$

$\mathbb{E}(f(X))^2 + \mathbb{E}(f(Y))^2 - 2\mathbb{E}f(X)\mathbb{E}f(Y) \leq 2\,\mathrm{Var}(X)$

since X, Y independent, $\mathbb{E}f(X) = \mathbb{E}f(Y)$

$\mathbb{E}(f(X))^2 + \mathbb{E}(f(X))^2 - 2(\mathbb{E}f(X))^2 \leq 2\,\mathrm{Var}(X)$

$2\,\mathrm{Var}(f(X)) \leq 2\,\mathrm{Var}(X)$

$\mathrm{Var}(f(X)) \leq \mathrm{Var}(X)$

End of exam