Math 170A - Winter 2021

Second Midterm Exam

Name:

1. **READ THE RULES FIRST**

2. Submit your answers to gradescope as our normal homework

3. Throughout the test, <u>**show your work so that your reasoning is clear**</u>. Otherwise no credit will be given.

4. Send me emails if you have questions. But in most cases, you will not get reply, please consider it as: **the exam is correct as it is**.

5. Help from other person, AI or **internet** are not allowed.

6. **Calculator** is allowed (needed) .

7. It will take about **1.5 hr - 2hr** to finish this exam.

Problem	Points	Grade
1	15	
2	15	
3	15	
4	10	
5	20	
6	10	
7	5	
8	5	
Total	100	

1. (15 pts, 5 pts. each)

    (a) Let X be a random variable with

$$p_X(k) = 1/7, \quad k \in [-3, 3], \quad k \in \mathbb{Z}$$

    Please find p.m.f p_Y, here $Y = (X - 2)^2$.

    (b) Calculate $\mathbb{E}e^{-(X+3)^2/10}$. (Your answer should have 2 significant figures).

    (c) Let Z be a random variable with $\mathbb{E}Z = 1$, $\mathbb{E}Z^2 = 3$, $\mathbb{E}Z^3 = 7$, $\mathbb{E}Z^4 = 20$. Please find $\mathrm{Var}(2Z^2 + 3)$.

2. (15 pts, 5 pts. each)

Let X be a geometric random variable with $\mathbb{E}X = 6$. We toss X coins independently. For each coin,
$$\mathbb{P}(Head) = p, \quad 0 < p < 1.$$
Let Y be the number of heads among these X coins.

(a) What is the $\mathbb{E}Y$? (Please simplify your answer for 2(a)).

(b) Similarly you toss X (fair 6 sided) dices and let Z be the number of 6's you get from these dices.

What is the $\mathbb{E}(Z^2)$? (Please simplify your answer for 2(b)).

(c) If $p = 1/2$, what is the $\mathbb{E}[Y^2|A]$, where $A = \{X \geq 2\}$.

Find your numerical answer with error less than 1, e.g., 8.3 ± 1 (Calculator is allowed).

3. (15 pts, 5 pts. each) Suppose that the joint probability mass function for X and Y is given by

$$p_{X,Y}(1,2) = 1/3$$
$$p_{X,Y}(1,1) = 1/4$$
$$p_{X,Y}(2,3) = 1/4$$
$$p_{X,Y}(3,3) = 1/6$$
$$p_{X,Y}(x,y) = 0, \qquad \text{for all other pairs } (x,y).$$

(a) Find $\mathbb{E}[XY + X + Y]$.

(b) Find $\mathbb{P}(X = Y - 1)$.

(c) Find $p_Y(y)$, i.e., the marginal probability mass function for Y.

4. (10 pts) After doing the laundry, you start to pair socks. You have 10 **different** pair of socks. You want to do it in the random way. You put all of them into a box. Randomly draw two socks at one time, if it is a pair, you put (tie) these two socks together and don't put it back to the box. Otherwise you put these two socks back to the box and try again.

(a) (6pts)

Assume that you tried X times to get the first pair of socks.
Write the probability mass function p_X **and** $\mathbb{E}X$.

(b) (4pts)

Let Y to be the total number of drawing for pairing all of them.
What is $\mathbb{E}Y$?

5. (25 pts, 5 pts. each) Randomly put 80 balls into 40 boxes. The balls are label from 1 to 80 and The balls are label from 1 to 40. The rule is: first you randomly pick a ball, put into a random box. Then randomly pick another ball, put into a random box (could be the same as the first one), again and again until these 80 balls are all in some box.

(a) Suppose the $k - th$ ball is in the X_k-th box. (e.g. $X_{20} = 17$ means the 20th ball is in the 17th box) Please find the probability mass function of X_k, i.e., p_{X_k}

(b) Let Y_k be the number of balls in the $k-$th box. Are events $\{X_1 = 1\}$ and $\{X_2 = 2\}$ conditionally independent given $\{Y_3 = 2\}$? Why ?

(c) What is $\mathbb{E}\left[Y_1 \middle| \{Y_2 = 0\}\right]$? Simplify your answer.

(d) Let Z be the total number of empty boxes, Please find $\mathbb{E}(Z)$.

Hint: $Z = \sum_k Z_k$. Here, $Z_k = 1$ if the $k - th$ box is empty.

(e) Please find $\text{Var}(Z)$. Hint: $\mathbb{E}Z_1 Z_2 = ?$

6. (10 pts)

A submarine lost at sea. There are two (and only two) possible regions: A and B. This question is about in which region people should search.

In the beginning, experts estimate $\mathbb{P}$ (submarine lost lost in region A) to be 70%. Furthermore, if it did lose in region A, for each search, the probability of actually finding the submarine is 40%. This number is 80% for region B.

Case A: (6 pts) So far, Navy has searched region A once, but did **not** find the submarine. Now based on these informations, in this case which region they should search next? And why?

Case B: (4 pts) So far, Navy has searched region A twice (with 2 teams independently), but did **not** find the submarine. Now based on these informations, in this case which region they should search next? And why?

Note: They are two different cases. They are "parallel" to each other.

7. (5pts)

An urn has 20 black and 20 white balls. We randomly remove balls, one at a time until the balls in the bag have the same colors to each other (or there is only 1 ball left in the bag).

Please calculate the expectation of the number of balls left in the bag eventually. Find your numerical answer with error less than 1. (Calculator is allowed).

Hint: you saw a similar question in midterm one, right.

8. (5pts)

Let X be a Poisson random variable with $EX = 5$, please show that it is $Poisson(5)$.