MATH 170A Final

John Arthur Minhquan Dang

TOTAL POINTS

89 / 100

QUESTION 1

Question 1 10 pts

1.1 (a) 5 / 5

✓ + **2 pts** **Recalls PDF of exponential distribution**

✓ + **2 pts** **Sets up integral correctly**

✓ + **1 pts** **Evaluates integral correctly**

1.2 (b) 5 / 5

✓ + **2 pts** **Knows formula for variance**

✓ + **2 pts** **Sets up integral(s) correctly**

✓ + **1 pts** **Evaluates correctly**

+ **1 pts** Just states the value of Var[X], with no other work. (Mutually exclusive with other points).

QUESTION 2

Question 2 10 pts

2.1 (a) 4 / 5

- **0 pts** Correct

✓ - **1 pts** **Forgets '0 otherwise'**

- **1 pts** Error in computing X mod 3

- **1 pts** Other small computational error

- **3 pts** Structural error

- **5 pts** Blank / no answer

2.2 (b) 5 / 5

- **0 pts** Correct

✓ - **0 pts** **Forgot '0 otherwise', but had already lost points for this in part (a)**

- **1 pts** Forgot '0 otherwise', had NOT already lost points for this in part (a)

- **1 pts** Error in computing 5 mod (X + 1)

- **1 pts** Other small computational error

- **3 pts** Structural error

- **5 pts** Blank / no answer

QUESTION 3

Question 3 15 pts

3.1 (a) 3 / 3

✓ + **3 pts** **Correct**

+ **2 pts** Small mistake

+ **1 pts** No significant progress towards a solution / solution with major structural problems

+ **0 pts** Blank / no answer

3.2 (b) 2.5 / 3

- **0 pts** Correct

✓ - **0.5 pts** **Forgets values outside of [0, 2] (should be zero for r < 0 and one for r > 2).**

- **1 pts** Sets up integral but is unable to proceed. (Switching to polar makes the integral simple).

- **1 pts** Computation error

- **2 pts** Not correct

- **3 pts** Blank/no answer

3.3 (c) 3 / 3

✓ - **0 pts** **Correct**

- **0 pts** Incorrect, but consistent from part (b)

- **0.5 pts** Forgets '0 otherwise' / domain issues

- **1 pts** Incorrect, but knows that the PDF is the derivative of the CDF

- **2 pts** Incorrect

- **3 pts** Blank / no answer

3.4 (d) 3 / 3

✓ - **0 pts** **Correct**

- **0 pts** Incorrect, but consistent from (c) and/or (b)

- **1 pts** Incorrect, but knows how to find the expected value from the PDF or CDF

- **2 pts** Incorrect

- **3 pts** Blank / no answer

3.5 (e) 2.5 / 3

- **- 0 pts** Correct
- ✓ **- 0.5 pts Domain issues (need zero outside the interval [-sqrt(3), sqrt(3)]**
- **- 1 pts** Computation mistake
- **- 2 pts** Incorrect, but some correct statement
- **- 3 pts** Blank / no answer

QUESTION 4

4 Question 4 6 / 6

- ✓ **+ 6 pts Correct**
- **+ 1 pts** Writes PMF of Poisson
- **+ 2 pts** Writes definition of expected value
- **+ 0 pts** Blank / no answer

QUESTION 5

5 Question 5 2 / 7

- **+ 7 pts** Correct
- ✓ **+ 2 pts Says that E[X_1 + ... + X_N] = E[X_1] + ... + E[X_N]. This is not correct; the left-hand side is a number and the right-hand side is a random variable... (The addition of expectation does not and should not be expected to work for a random number of terms). Or, says that X_1 + ... + X_N is NX_1. This is the opposite of iid. Also, it doesn't make sense to write NE[X_1] = E[X_1N] ... one of these is a RV, and the other a number. Also, it should be pointed out that expectations sum even if the random variables are not independent.**
- **+ 1 pts** No significant progress towards a solution.
- **+ 0 pts** Blank/no answer

QUESTION 6

Question 6 10 pts

6.1 (a) 5 / 5

- ✓ **- 0 pts Correct**
- **- 1 pts** Computational mistake
- **- 2 pts** Does not take into account that there are 100 employees (and so finds - correctly - the probability that one employee is between 70k and 80k)
- **- 3 pts** Incorrect, some correct ideas
- **- 5 pts** Blank / no answer

6.2 (b) 5 / 5

- ✓ **- 0 pts Correct**
- **- 1 pts** Error in computing standard deviation of average (it is 500)
- **- 1 pts** Error in using normal table
- **- 3 pts** Incorrect, recognizes normal distribution of average
- **- 4 pts** Incorrect, no substantial progress towards a solution
- **- 5 pts** Blank / no answer

QUESTION 7

7 Question 7 7 / 7

- ✓ **+ 7 pts Correct**
- **+ 0 pts** Incorrect
- **+ 1 pts** LoTE for E[X]
- **+ 3 pts** LoTE for E[X|T_1]
- **+ 3 pts** LoTE for E[X|H_1]

QUESTION 8

Question 8 10 pts

8.1 (a) 4 / 4

- ✓ **+ 4 pts Correct**
- **+ 0 pts** Incorrect

8.2 (b) 3 / 3

- ✓ **+ 3 pts Correct**
- **+ 0 pts** Incorrect

8.3 (c) 3 / 3

- ✓ **+ 3 pts Correct**
- **+ 0 pts** Incorrect

QUESTION 9

9 Question 9 4 / 6

- ✓ **+ 6 pts Correct**
- **+ 0 pts** Incorrect
- **- 2 Point adjustment**

Should have had two integrals and broken integral up around c for dx.

QUESTION 10

10 Question 10 **4 / 6**

✓ **+ 6 pts** **Correct**

+ 0 pts Incorrect

- 2 **Point adjustment**

P(X^2 + Y^2 >=1) =/= 1 - P(-1 <= X <= 1)P(-1 <= Y <= 1)

QUESTION 11

11 Question 11 **7 / 7**

✓ **+ 7 pts** **Correct**

+ 0 pts Incorrect

QUESTION 12

12 Question 12 **6 / 6**

✓ **+ 6 pts** **Correct**

+ 0 pts Incorrect

Answer the questions in the spaces provided. If you are using pencil, please write as darkly as possible. Explain your answers clearly - if the answer requires computation, write it down. Unless otherwise noted, please simplify. No phones or calculators are allowed. Only 2 pages (front and back) of notes are allowed. Sign your name below - by doing so you are agreeing to abide by the UCLA Student Code of Conduct.

Name: John Arthur Dang

Student ID Number: [redacted]

Signature:

Table 1: Table of the Standard Normal Cumulative Distribution Function $\Phi(z)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

1. Consider a random variable, X, distributed exponentially with parameter λ; i.e., $X \sim \exp(\lambda)$.

(a) Compute the mean of X, $\mathbb{E}(X)$.

$$f_X(x) = \lambda e^{-\lambda x}$$

Integration by parts
$u = x \quad dv = e^{-\lambda x}dx$
$\frac{du}{dx} = 1 \quad v = \frac{1}{-\lambda}e^{-\lambda x}$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \lambda\int_0^{\infty} x e^{-\lambda x}\,dx$$

$$= \lambda\left(-\frac{x}{\lambda}e^{-\lambda x}\Big|_0^{\infty} - \int_0^{\infty} -\frac{1}{\lambda}e^{-\lambda x}\,dx\right)$$

$$= \lambda\left(0 + \frac{1}{\lambda}\left[\frac{1}{-\lambda}e^{-\lambda x}\right]_0^{\infty} dx\right)$$

$$= \lambda\left(\frac{1}{\lambda}\left[\frac{1}{\lambda}\right]\right)$$

$$= \lambda\left(\frac{1}{\lambda^2}\right) = \frac{1}{\lambda}$$

(b) Compute the variance of X, $\mathrm{var}(X)$.

$$\mathrm{var}(X) = E[X^2] - (E[X])^2$$

$$= \int_{-\infty}^{\infty} x^2 f_X(x)\,dx - \left(\frac{1}{\lambda}\right)^2$$

$$= \lambda\int_0^{\infty} x^2 e^{-\lambda x}\,dx - \frac{1}{\lambda^2}$$

$$= \frac{x^2}{-\lambda}e^{-\lambda x} + \frac{1}{\lambda}\left[\frac{2}{-\lambda}e^{-\lambda x} + \frac{2}{-\lambda^2}e^{-\lambda x}\right]\Big|_0^{\infty} - \frac{1}{\lambda^2}$$

$$= \frac{1}{\lambda^2}$$

Integration by parts twice
$u = x^2 \quad dv = e^{-\lambda x}dx$
$\frac{du}{dx} = 2x \quad v = \frac{1}{-\lambda}e^{-\lambda x}$

$$\frac{x^2}{-\lambda}e^{-\lambda x} + \frac{1}{\lambda}\int e^{-\lambda x}2x\,dx$$

$u = 2x \quad dv = e^{-\lambda x}dx$
$\frac{du}{dx} = 2 \quad v = -\frac{1}{\lambda}e^{-\lambda x}$

$$\frac{x^2}{-\lambda}e^{-\lambda x} + \frac{1}{\lambda}\left[\frac{2x}{-\lambda}e^{-\lambda x} + \frac{2}{\lambda}\int e^{-\lambda x}\,dx\right]$$

2. Let X be a random variable that takes values from 0 to 9 with equal probability 1/10.

(a) Find the PMF of the random variable $Y = X \mod (3)$.

X	Y
0	0
1	1
2	2
3	0
4	1
5	2
6	0
7	1
8	2
9	0

$$P_Y(y) = \begin{cases} \frac{1}{10}(4), & y=0 \\ \frac{1}{10}(3), & y=1 \\ \frac{1}{10}(3), & y=2 \end{cases} = \begin{cases} \frac{4}{10}, & y=0 \\ \frac{3}{10}, & y=1 \\ \frac{3}{10}, & y=2 \end{cases}$$

(b) Find the PMF of the random variable $Y = 5 \mod (X+1)$.

X	Y
0	0
1	1
2	2
3	1
4	0
5	5
6	5
7	5
8	5
9	5

$$P_Y(y) = \begin{cases} 2\left(\frac{1}{10}\right), & y=0 \\ 2\left(\frac{1}{10}\right), & y=1 \\ 1\left(\frac{1}{10}\right), & y=2 \\ 5\left(\frac{1}{10}\right) & y=5 \end{cases} = \begin{cases} \frac{2}{10}, & y=0 \\ \frac{2}{10}, & y=1 \\ \frac{1}{10}, & y=2 \\ \frac{5}{10}, & y=5 \end{cases}$$

3. Suppose you are to throw a dart at a circular dart board with radius 2 inches. Let (X, Y) denote the point that you hit on the board (you can assume the board is centered at the origin $(0, 0)$, and that the dart hits somewhere on the board uniformly at random).

(a) Define $C_r := \{(x, y) | x^2 + y^2 = r^2\}$. Show that the probability that you hit a point at any given fixed distance from the center is 0; i.e., $\mathbb{P}((X, Y) \in C_r) = 0$ for all $r > 0$.

$f_{X,Y}(X,Y) = \frac{1}{\text{area of circle}} = \frac{1}{\pi r^2} = \frac{1}{\pi(2^2)} = \frac{1}{4\pi}$

Using polar coordinates: $X = K\cos\theta$, $Y = K\sin\theta$

$P((X,Y) \in C_r) = \int_0^{2\pi} \int_r^r f_{X,Y}(x(K,\theta), y(K,\theta))\, K\, dK\, d\theta$

$= \int_0^{2\pi} 0 \, d\theta = 0$

(b) Let $R = \sqrt{X^2 + Y^2}$ so that $R \geq 0$. Compute the CDF of R, $F_R(r)$.

$P(R \leq r) = \frac{\text{area of circle radius } r}{\text{area of circle radius } 2}$ for $0 \leq r < 2$

$= \frac{\pi r^2}{4\pi} = \frac{r^2}{4}$

$F_R(r) = \begin{cases} \frac{r^2}{4}, & 0 \leq r < 2 \\ 1, & 2 \leq r \end{cases}$

(c) Compute the PDF of R, $f_R(r)$.

$$f_R(r) = \frac{d}{dr} F_R(r) = \frac{d}{dr}\left[\frac{r^2}{4}\right] = \frac{r}{2}$$

$$= \begin{cases} \frac{r}{2}, & 0<r<2 \\ 0, & \text{otherwise} \end{cases}$$

(d) Find the expected distance between (X, Y) and the origin.

The distance between (X,Y) and the origin is the random variable R from earlier

$$E[R] = \int_0^2 r\, f_R(r)\, dr = \int_0^2 \frac{r^2}{2} = \frac{r^3}{6}\Big|_0^2$$

$$= \frac{8}{6} = \frac{4}{3}$$

(e) Suppose you know $X = 1$. What is the PDF of Y conditioned on this fact?

$$f_{Y|X}(Y|X=1) = \frac{f_{X,Y}(x,y)}{f_X(1)}$$

$$f_X(1) = \int_{-\sqrt{3}}^{\sqrt{3}} f_{X,Y}(x,y)\, dy = \frac{2\sqrt{3}}{4\pi}$$

$$f_{Y|X}(Y|X=1) = \frac{\frac{1}{4\pi}}{\frac{2\sqrt{3}}{4\pi}} = \frac{1}{2\sqrt{3}}$$

4. A baseball team loses $100,000 for each consecutive day it rains. Say X, the number of consecutive days it rains at the beginning of the season, has a Poisson distribution with mean 0.2. What is the expected loss before the opening game?

$$P_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

$\lambda = 0.2$ since $E[X] = \lambda = 0.2$

$$= \frac{e^{-0.2}(0.2)^k}{k!}$$

Let Y = loss before opening game

$$Y = 100000 \cdot X$$

$$E[Y] = E[X]\,100000$$

$$= 0.2\,(100000)$$

$$= \$20{,}000$$

by linearity of expectation

5. Let N be a nonnegative integer-valued random variable, and let X_i be a sequence of independent and identically distributed random variables. Assume $\mathbb{E}[X_i]$ and $\mathbb{E}[N]$ are finite. Show that $\mathbb{E}[X_1 + X_2 + ... + X_N] = \mathbb{E}[X_1]\mathbb{E}[N]$.

$E[X_1 + X_2 + ... X_N]$

① $= E[X_1] + E[X_2] + ... + E[X_N] = \sum_{i=1}^{N} E[X_i]$

by linearity of expectation

Since X_i are all identically distributed and independent we have $E[X_1] = E[X_i]$

Thus we get

② $= E[X_1] E[N]$

since the expected # of experiments summed in step 1 is $E[N]$

6. A company pays its junior employees salaries which are approximately normally distributed with a standard deviations of \$5,000 and a mean of \$75,000.

   (a) At a specific branch, there are 100 junior employees. Assuming the salaries are independent and identically distributed according to the distribution above, what is the probability that the maximum salary and the minimum salary are both within \$5,000 of the mean? Hint: Use the CDF table to compute the probability that one salary is within \$5,000 of the mean and then extend this to the maximum and minimum salaries (you don't need to simplify the exponential at the end).

$\mu = 75000 \quad \sigma = 5000$

P(min and max within 5000 of mean)

= P(all employees make between 70000 and 80000)

= [P(one employee makes between 70K and 80K)]^{100}

since employee salaries are independent and identically distributed and the multiply normalizing to standard normal $z_a = \frac{80K-75K}{5K} = 1 \quad z_b = \frac{70K-75K}{5K} =$

$= [P(z \le z_a) - P(z \le z_b)]^{100} = [\Phi(1) - \Phi(-1)]^{100} = (0.8413 - 0.1587)^{100}$

   (b) At a specific branch, there are 100 junior employees. Assuming the salaries are independent and identically distributed according to the distribution above, what is the probability that the average salary is within \$1,000 of the mean?

Let X_n = salary of n^{th} employee for $n = 1, \dots, 100$

Let A = average salary

$A = \frac{1}{100} \sum_{n=1}^{100} X_n$

X_n are normal i.i.d. so variance adds

$\sigma_A^2 = Var\left(\frac{1}{100}\sum_{n=1}^{100} X_n\right) = \sum_{n=1}^{100} var(X_n) = \frac{1}{100^2}\sum_{n=1}^{100}(5000)^2 = \frac{1}{100}(5000)^2$

mean (Expectation) is linear so $\mu_A = \frac{1}{100}\sum_{n=1}^{100} \mu_{X_n} = 75000$

A is normal with mean 75000 and variance $\frac{5000^2}{100}$

Normalizing to standard normal $\sigma = \frac{5000}{\sqrt{100}} = \frac{5000}{10} = 500$

$z_a = \frac{76000 - 75000}{500} = 2 \quad z_b = \frac{74000 - 75000}{500} = -2$

P(ave salary within 1000) $= P(z \le 2) - P(z \le -2) = \Phi(2) - \Phi(-2) = 0.9772 - 0.0228$

7. Suppose you flip a fair coin repeatedly until you see a Tails followed by a Heads. What is the expected number of coin flips you have to flip?

Let X = # of coin flips

By law of total expectation

$E[X] = \frac{1}{2}E[X|H_1] + \frac{1}{2}E[X|T_1]$

$E[X|H_1] = \frac{1}{2}E[X|H_1T_2] + \frac{1}{2}E[X|H_1H_2]$

$E[X|H_1] = \frac{1}{2}(1+E[X|T_1]) + \frac{1}{2}(1+E[X|H_1])$

$E[X|H_1] = 1 + \frac{1}{2}E[X|T_1] + \frac{1}{2}E[X|H_1]$

$\frac{1}{2}E[X|H_1] = 1 + \frac{1}{2}E[X|T_1]$

$E[X|H_1] = 2 + E[X|T_1]$

$E[X|T_1] = \frac{1}{2}E[X|T_1H_2] + \frac{1}{2}E[X|T_1T_2]$

$E[X|T_1] = \frac{1}{2}(2) + \frac{1}{2}(1+E[X|T_1])$

$E[X|T_1] = 1 + \frac{1}{2} + \frac{1}{2}E[X|T_1]$

$E[X|T_1] = 3$

$E[X|H_1] = 2+3 = 5$

$E[X] = \frac{1}{2}(5) + \frac{1}{2}(3)$

$= 4$

Ⓡ Ⓡ Ⓑ

8. Suppose 3 players are playing a game in which they sequentially draw one of 3 balls from a bag. Two of the balls are red and one is blue, and the first player who draws the blue ball from the bag is the winner.

   (a) If each player draws one ball from the bag at random and *doesn't replace the ball*, compute the probability that the third player wins the game.

$$P(\text{Third wins}) = P(\text{first red})\,P(\text{second red} \mid \text{first red})$$
$$= \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$$

   (b) If each player draws one ball from the bag at random and *replaces the ball* (each player gets only one draw), compute the probability that the third player wins the game.

$$P(\text{Third wins}) = P(\text{first red})\,P(\text{second red})$$
$$= \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{27}$$

   (c) If each player draws one ball from the bag at random and *replaces the ball* (each player gets only one draw), compute the probability that no player wins the game.

$$P(\text{no winner}) = P(\text{first red})\,P(\text{second red})\,P(\text{Third red})$$
$$= \frac{2}{3}\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \frac{8}{27}$$

9. Suppose the joint PDF of random variables X and Y, $f_{X,Y}(x,y)$, is symmetric about the line $x = c$; that is $f_{X,Y}(c+t,y) = f_{X,Y}(c-t,y)$ for all $t \geq 0$ and all y. Prove that the marginal CDF of X satisfies $F_X(c) = 0.5$.

$$\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = 1$$

$$\int_{-\infty}^{c} f_{X,Y}(x,y)\,dy + \int_{c}^{\infty} f_{X,Y}(x,y)\,dy = 1$$

Since $f_{X,Y}(c+t,y) = f_{X,Y}(c-t,y)$

we have $\int_{-\infty}^{c} f_{X,Y}(x,y)\,dy = \int_{c}^{\infty} f_{X,Y}(x,y)\,dy$

By substitution we get

$$2\int_{-\infty}^{c} f_{X,Y}(x,y)\,dy = 1$$

$$\int_{-\infty}^{c} f_{X,Y}(x,Y)\,dy = 0.5$$

$$F_X(c) = \int_{-\infty}^{c} f_{X,Y}(x,y)\,dy = 0.5$$

10. The coordinates X and Y of a point are independent standard normal random variables. Given that the point is at a distance of at least 1 from the origin, find the conditional joint PDF of X and Y.

Condition of at least 1 from the origin implies $x^2+y^2 \geq 1$

standard normal independent means

$$\mu_X = \mu_Y = 0$$

$$\sigma_X^2 = \sigma_Y^2 = 1$$

Let A be the event that $x^2+y^2 \geq 1$

$$f_{X,Y|A}(x,y) = \frac{f_{X,Y}(x,y)}{P(A)} = \frac{f_X(x)\, f_Y(y)}{P(A)}$$

Since X and Y are independent

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

$$P(A) = 1 - P(-1 \leq X \leq 1 \cap -1 \leq Y \leq 1)$$

since X, Y independent

$$= 1 - (\Phi(1) - \Phi(-1))^2$$

$$= 1 - (0.8143 - 0.1587)^2$$

$$f_{X,Y|A}(x,y) = \frac{1}{2\pi\left[1-(0.8143-0.1587)^2\right]} e^{-\frac{x^2}{2}-\frac{y^2}{2}}$$

11. Suppose two independent claims are made on two insured homes, where each claim has pdf

$$f_X(x) = \frac{4}{x^5}, \quad 1 < x < \infty,$$

in which the units are \$1000. Find the expected value of the larger claim.
Hint: If X_1 and X_2 are the two independent claims and $Y = \max(X_1, X_2)$ then

(1) $F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(X_1 \le y)\mathbb{P}(X_2 \le y) = [\mathbb{P}(X \le y)]^2$.

$Y = \max(X_1, X_2)$

$F_Y(y) = (P(X \le y))^2 = \left(\int_1^y \frac{4}{x^5}\,dx\right)^2$

$= \left(\frac{-1}{x^4}\Big|_1^y\right)^2$

$= \left(\frac{-1}{y^4} + 1\right)^2$

$f_Y(y) = \frac{d}{dy}\left(\frac{-1}{y^4}+1\right)^2 = 2\left(-\frac{1}{y^4}+1\right)\frac{4}{y^5} = -\frac{8}{y^9} + \frac{8}{y^5}$

$E[Y] = \int_1^\infty y\left(-\frac{8}{y^9} + \frac{8}{y^5}\right) dy = -8\int_1^\infty \frac{1}{y^8} - \frac{1}{y^4}\, dy$

$= -8\left[-\frac{1}{7y^7} + \frac{1}{3y^3}\right]_1^\infty$

$= -8\left[(0) - \left(-\frac{1}{7} + \frac{1}{3}\right)\right]$

$= 8\left(\frac{1}{3} - \frac{1}{7}\right) = 8\left(\frac{7}{21} - \frac{3}{21}\right) = \frac{32}{21} = \$\frac{32}{21}(1000)$

12. You go to a party with 500 other guests. What is the probability that exactly one other guest has the same birthday as you? For simplicity, ignore leap years and assume that every year has exactly 365 days with a given birthday equally likely to be on any of these days.

P(1 other guest has same birthday)

$$= \binom{500}{1}\frac{1}{365}\left(\frac{364}{365}\right)^{499}$$

since P(your birthday) $= \frac{1}{365}$

and P(not your birthday) $= 1 - \frac{1}{365} = \frac{364}{365}$

binomial with $n = 500$ and $p = \frac{1}{365}$