Math 170A: Probability Theory, A. Iseli 27 January 2020

Midterm 1 (Duration: 50 min)

Name and UID: [redacted]

By signing the line below, you affirm that you did not cheat on this exam and that you understand and accept the following conditions.

- Do not open the exam until instructed to do so.
- **No phones, calculators, books, or notes are permitted.**
- Nothing should be on your desk but writing implements.
- If you have a question during the exam, please raise your hand.
- Only write your solutions on the **front side of the numbered pages**. There is an extra page at the end of the exam if you need more space.
- Please write neatly, show all your work, and **justify all answers**. Mysterious or illegible solutions will receive no credit.
- In case you finish early, you may hand in your exam and leave the room quietly. However, please don't hand in or leave during the last 20 minutes of the exam.
- You may be recorded by photo or video to ensure testing integrity.
- This exam may be graded using gradescope.

No exam without signature will be graded!

Signature: ____________________

Question	Points	Score
1	9	
2	7	
3	6	
4	10	
Total:	32	

1. Three short (not interrelated) problems.

(a) (3 points) Let $\mathcal{F}$ and $\mathcal{B}$ be two event spaces for the same sample space Ω. Prove that $\mathcal{F} \cap \mathcal{B}$ is an event space.

① $\phi, \Omega \in F, B \rightarrow \phi, \Omega \in F \cap B \rightarrow F \cap B \neq \phi$.

② Let $A \in F \cap B \rightarrow \left\{ \begin{array}{l} A \in F \Rightarrow A^c \in F \\ A \in B \Rightarrow A^c \in B \end{array} \right\} \Rightarrow A^c \in F \cap B$

③ Let $A_1, \ldots \in F \cap B$. Then $\bigcup_{i \in \mathbb{N}} A_i \in F$ and $\bigcup_{i \in \mathbb{N}} A_i \in B$ because F & B are event spaces $\Rightarrow \bigcup_{i \in \mathbb{N}} A_i \in F \cap B$.

(b) (3 points) Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $A, B, C \in \mathcal{F}$. Assume that A and B are independent, A and C are independent, C and B are disjoint. Prove that A and $B \cup C$ are independent.

(Point out in your working where you use each of the assumptions.)

$P(A \cap B) = P(A)\,P(B)$
$P(A \cap C) = P(A)\,P(C)$
$P(B \cap C) = 0$.

$P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C)) = P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$
$= P(A)P(B) + P(A)P(C) - 0$.
$= P(A)(P(B) + P(C)) = P(A)\,P(B \cup C)$ (because B & C disjoint)

So A & $B \cup C$ independent.

(c) (3 points) Let $\Omega = \{1, \ldots, n\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $p_1, \ldots, p_n > 0$ so that $\sum_{i=1}^{n} p_i = 1$. Let $P : \mathcal{F} \to \mathbb{R}$ be a function defined by $P(A) := \sum_{i \in A} p_i$ for all $A \in \mathcal{F}$. Prove that P is a probability measure on $(\Omega, \mathcal{F})$.

① $\forall A \in F$, $P(A) = \sum_{i \in A} p_i \geq 0$ because $p_i \geq 0 \ \forall i \in \{1, \ldots, n\}$ ✓

② $P(\Omega) = \sum_{i \in \Omega} p_i = \sum_{i=1}^{n} p_i = 1$ ✓

③ Let $A_1, \ldots, A_m$ be disjoint sets in F. (finite because Ω is finite).

Then $\sum_{i=1}^{m} P(A_i) = \sum_{j \in A_1} p_j + \sum_{j \in A_2} p_j + \cdots + \sum_{j \in A_m} p_j = \sum_{i=1}^{m} \sum_{j \in A_i} p_j$

There exists no j such that $p_j \in A_{i_1} \cap A_{i_2}$. So $\sum_{i=1}^{m} \sum_{j \in A_i} p_j = \sum_{j \in A_i} p_j$ for some i 1,...,m

$\rightarrow = \sum_{j \in \bigcup_{i=1}^{m} A_i} p_j = P\left(\bigcup_{i=1}^{m} A_i\right)$.

2. You missed class and so you don't know which chapters will be part of Monday's exam. The only way to find out is asking one of your 36 classmates. Among those, 10 are serious and trustworthy people and will give you the correct answer for sure; 24 never pay attention in class and therefore will give you a correct answer with a probability of 75%; and 2 are notorious liars and will give you the wrong answer for sure.

   (For this problem, you are NOT required to specify the underlying probability space. However, you should point out in your working the theorems that you apply.)

   (a) (3 points) If you ask one classmate at random, what is the probability that you will obtain a correct answer?

   (b) (1 point) Are the two following events independent?

      - You obtain a correct answer.
      - You ask one of the notorious liars.

   (c) (3 points) After the exam, you notice that you were given an incorrect answer. What is the probability that you had asked one of the notorious liars?

(a) Partition theorem: $P(A) = \sum P(A|B_i)\,P(B_i)$

A = correct answer
B_i = different types of people
B_1 = trustworthy
B_2 = don't pay attention
B_3 = liar

$\frac{10}{36}\cdot(1) + \frac{24}{36}\cdot\left(\frac{3}{4}\right) + \frac{2}{36}(0) = \frac{10}{36} + \frac{18}{36} = \frac{28}{36} = \boxed{\frac{7}{9}}$

(b) P(correct answer) = $7/9$

P(liar) = $2/36 = 1/18$

P(correct answer & liar) = 0 $\neq$ P(correct answer) * P(liar)

So $\boxed{\text{no}}$ they are not independent

(c)

Let A = event that you get incorrect answer

Bayes Thm

$$P(B_3 | A) = \frac{P(A|B_3)\,P(B_3)}{\sum_{i=1}^{3} P(A|B_i)\,P(B_i)} = \frac{\frac{2}{36}\cdot 1}{\frac{10}{36}\cdot 0 + \frac{24}{36}\cdot\left(\frac{1}{4}\right) + \frac{2}{36}\cdot(1)} = \frac{1/18}{1/6 + 1/18} = \frac{1}{18}\cdot\frac{18}{4}$$

$= \boxed{\frac{1}{4}}$

3. (a) (3 points) Give the definition of a discrete random variable on a probability space.
   (b) (3 points) Give an example of a discrete random variable on a probability space whose values include $E := \{1, 2, 7\}$ and each $a \in E$ is attained with probability $\frac{1}{5}$

(a) Let (Ω, F, P) be a probability space.

A discrete RV X on (Ω, F, P) is a function

$X : \Omega \to \mathbb{R}$ such that

$X(\Omega) = \{ x \in \mathbb{R} : \exists\, w \in \Omega \text{ s.t. } X(w) = x \}$ is countable

and $\forall a \in \mathbb{R}$, $X^{-1}(\{a\}) = \{ w \in \Omega : X(w) = a \} \in F$.

(b) $E = \{1, 2, 7\} \subseteq X(\Omega)$ $p_X(1) = p_X(2) = p_X(7) = 1/5$.

Define: $\Omega = \{1, 2, 3, 4, 5\}$

$F = P(\Omega)$

$P(A) = \dfrac{|A|}{|\Omega|} = \dfrac{|A|}{5}$

Define

$X : \Omega \to \mathbb{R}$ by $X(y) = \begin{cases} 1 & \text{if } y = 1 \\ 2 & \text{if } y = 2 \\ 7 & \text{if } y = 3 \\ 10 & \text{if } y = 4 \\ 11 & \text{if } y = 5. \end{cases}$

4. (10 points) **True or false.**

   For each of the statements below, decide whether it is true or false. Moreover, if it is true give a <u>short</u> explanation or proof (one or two sentences); if it is false, give a counterexample (without proof).

   Let $(\Omega, \mathcal{F}, P)$ be a probability space.

False (a) For every two events $A, B \in \mathcal{F}$, with $P(B) \in (0,1)$ we have $P(A|B)+P(A|B^c) = 1$.

Let $\Omega = \{1,2,3,4,5\}$ $\quad F = \mathcal{P}(\Omega)$ $\quad P(A) = \frac{|A|}{|\Omega|}$

$A = \{1,2\}$ $\quad B = \{1,2,3\}$ $\rightarrow$ $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{2}{3}$

So $P(A|B) + P(A|B^c) = \frac{2}{3} + 0 = \frac{2}{3} \neq 1$. $\quad P(A|B^c) = \frac{P(A\cap B^c)}{P(B^c)} = 0$

False (b) If $P(A) = 0$ for some $A \in \mathcal{F}$, then $A = \varnothing$.

$\Omega = \{1,2,3\}$ $\quad F = \mathcal{P}(\Omega)$ $\quad P(\{1\}) = 0$ $\quad P(\{2\}) = 1/2$ $\quad P(\{3\}) = 1/2$ $\quad P(A) = \sum_{x\in A} P(\{x\})$

① $\forall A \in F$, $P(A) \geq 0$ ② $P(\Omega) = P(\{1\}) + P(\{2\}) + P(\{3\}) = 0 + 1/2 + 1/2 = 1$

③ if A_i are disjoint, $P(\cup A_i) = \sum P(A_i)$. can check easily.

False (c) If A and B are independent events in $\mathcal{F}$, then $A \neq B$.

Counterexample: let (Ω, F, P) be a prob. space.

Let $A = B = \Omega$. Then $P(A\cap B) = P(\Omega) = 1$ and $P(A)P(B) = P(\Omega)P(\Omega) = 1$ $\Rightarrow$ A & B independent, $A = B$.

True (d) For all $A \in \mathcal{F}$, A and Ω are independent.

$P(A\cap\Omega) = P(A)$ because $A \cap \Omega = A$

$= P(A)\cdot 1 = P(A)P(\Omega)$ $\rightarrow$ independent.

False (e) For $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{F}$, we have $P\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} P(A_i)$.

Counterexample:

Let $\Omega = \{1,2,3,4,5\}$

$F = \mathcal{P}(\Omega)$

$P(A) = \frac{|A|}{|\Omega|}$

Let $A_1 = \{1,2,3\}$ $\quad A_2 = \{3,4,5\}$, $\quad A_i = \phi$ for $i \geq 3$.

Then $P(\bigcup_{i\in\mathbb{N}} A_i) = P(\{1,2,3,4,5\}) = P(\Omega) = 1$

$\sum_{i\in\mathbb{N}} P(A_i) = P(A_1) + P(A_2) + 0 + 0 + \cdots = \frac{3}{5} + \frac{3}{5} + 0 = \frac{6}{5}$ $\Rightarrow P(\bigcup_{i\in\mathbb{N}} A_i) \neq \sum_{i\in\mathbb{N}} P(A_i)$

(Extra space two write your solutions.)