Math 170A Final

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TOTAL POINTS

107 / 120

QUESTION 1

Question 1 18 pts

1.1 Part a 2 / 2

✓ **- 0 pts** Correct

- 2 pts Incorrect

1.2 Part b 2 / 2

✓ **- 0 pts** Correct

- 2 pts Incorrect

- 1 pts Blank

1.3 Part c 0 / 2

- 0 pts Correct

✓ **- 2 pts** Incorrect

- 1 pts Blank

1.4 Part d 2 / 2

✓ **- 0 pts** Correct

- 1 pts Blank

- 2 pts Incorrect

1.5 Part e 2 / 2

✓ **- 0 pts** Correct

- 1 pts Blank

- 2 pts Incorrect

1.6 Part f 2 / 2

✓ **- 0 pts** Correct

- 2 pts Incorrect

- 1 pts Blank

1.7 Part g 2 / 2

✓ **- 0 pts** Correct

- 2 pts Incorrect

- 1 pts Blank

1.8 Part h 0 / 2

- 0 pts Correct

✓ **- 2 pts** Incorrect

- 1 pts Blank

1.9 Part i 0 / 2

- 0 pts Correct

✓ **- 2 pts** Incorrect

- 1 pts Blank

QUESTION 2

Question 2 30 pts

2.1 Part a 5 / 5

✓ **- 0 pts** Correct

- 5 pts No solution to 2(a) submitted.

- 1 pts Range

- 1 pts Missing parameter

- 3 pts Wrong RV.

- 4 pts Click here to replace this description.

2.2 Part b 5 / 5

✓ **- 0 pts** Correct

- 5 pts No solution to 2(b) submitted.

- 1 pts Expected value

- 1 pts Range

- 1 pts Variance

- 1 pts Parameters

- 1 pts PMF

2.3 Part c 5 / 5

✓ **- 0 pts** Correct

- 5 pts No solution for 2(c) submitted

- 1 pts Variance

- 1 pts Range

- 1 pts Parameter

- **5 pts** Click here to replace this description.
- **1 pts** PMF
- **1 pts** Variance

2.4 Part d **5 / 5**

✓ **- 0 pts Correct**

- **1 pts** Variance
- **5 pts** No solution to 2(d) submitted.
- **1 pts** Expected value
- **1 pts** Range
- **5 pts** Click here to replace this description.
- **1 pts** Variance
- **1 pts** PDF
- **1 pts** Parameters

2.5 Part e **5 / 5**

✓ **- 0 pts Correct**

- **5 pts** No solution for 2(e) submitted
- **1 pts** Range
- **1 pts** Parameters
- **1 pts** PDF
- **3 pts** Click here to replace this description.
- **3 pts** Wrong Random Variable
- **1 pts** Variance

2.6 Part f **4 / 5**

- **0 pts** Correct
- **1 pts** Parameter Incorrect

✓ **- 1 pts Range Incorrect**

- **0 pts** Click here to replace this description.
- **5 pts** No submission
- **1 pts** distribution incorrect

QUESTION 3

Question 3 10 pts

3.1 Part a **4 / 5**

- **0 pts** Correct
- **5 pts** No solution to 3(a) provided.

✓ **- 1 pts Omitted countable additivity**

- **1 pts** Error
- **2 pts** Omitted additivity
- **3 pts** Missing normalization and incorrect statements of additivity.
- **3 pts** Missing normalization and additivity.
- **2 pts** Incorrect notation.

You gave a version of finite additivity.

3.2 Part b **5 / 5**

✓ **- 0 pts Correct**

- **5 pts** No submission

QUESTION 4

Question 4 10 pts

4.1 Part a **4 / 5**

- **0 pts** Correct

✓ **- 1 pts Overcounting**

- **1 pts** Undercounting
- **2 pts** Incorrect, unclear argument
- **1 pts** Incorrect, repetition in letters/numbers allowed.
- **2 pts** Incorrect, repetition allowed and have to take into account placement of letters/numbers

Need to use {6 choose 2} to choose locations of letters, not 6!

4.2 Part b **5 / 5**

✓ **- 0 pts Correct**

- **1 pts** Double counted
- **2 pts** Incorrect reasoning
- **4 pts** Incorrect with no justification or argument.
- **3 pts** Incorrect, unclear justification.

QUESTION 5

Question 5 10 pts

5.1 Part a **5 / 5**

✓ **- 0 pts Correct**

- **1 pts** Sign error
- **1 pts** Algebraic error
- **5 pts** no credit
- **4 pts** Slight progress

5.2 Part b 5 / 5

- ✓ - **0 pts** Correct
- - **1 pts** didn't switch minus to plus
- - **1 pts** didn't square coefficients
- - **1 pts** didn't eliminate constant
- - **1 pts** one variance wrong

QUESTION 6

Question 6 15 pts

6.1 Part a 5 / 5

- ✓ - **0 pts** Correct
- - **5 pts** No attempt

6.2 Part b 5 / 5

- ✓ - **0 pts** Correct
- - **1 pts** No domain for PDF specified
- - **5 pts** No credit
- - **1 pts** No value of C
- - **1 pts** Incorrect domain
- - **2 pts** Incorrect integral bounds

6.3 Part c 5 / 5

- ✓ - **0 pts** Correct
- - **5 pts** No credit
- - **4 pts** Correct formula for expectation
- - **4 pts** No correct working
- - **2 pts** Incorrect integral bounds
- - **1 pts** Incorrect PDF, correct calculations
- - **1 pts** numerical slip

QUESTION 7

Question 7 12 pts

7.1 Part a 5 / 5

- ✓ - **0 pts** Correct
- - **5 pts** no submission
- - **4 pts** some progress

7.2 Part b 4 / 5

- - **0 pts** Correct
- - **4 pts** Some progress
- ✓ - **1 pts** Close
- - **5 pts** No submission
- - **3 pts** non-trivial progress

7.3 Part c 2 / 2

- ✓ - **0 pts** Correct
- - **1 pts** Unclear somewhat correct answer
- - **2 pts** Incorrect

QUESTION 8

Question 8 15 pts

8.1 Part a 3 / 5

- - **0 pts** Correct
- - **5 pts** No solution to 8(a) given
- ✓ - **1 pts** X needs to be nonnegative
- ✓ - **1 pts** a needs to be >0
- - **1 pts** Mistake in inequality

8.2 Part b 4 / 5

- - **0 pts** Correct
- - **5 pts** No solution to 8(b) given
- ✓ - **1 pts** Specify a>0
- - **1 pts** Specify c>0
- - **1 pts** Mistake in inequality
- - **2 pts** Mistakes in inequality
- - **3 pts** Incorrect
- - **1 pts** P not E.

8.3 Part c 5 / 5

- ✓ - **0 pts** Correct
- - **5 pts** No solution for 8(c) submitted.
- - **1 pts** Minor error
- - **2 pts** Using inequalities incorrectly
- - **2 pts** Invalid reasoning
- - **3 pts** Unclear/incorrect reasoning

1. For each of the following statements, indicate whether they are True or False. **A blank answer will receive 1 point.** [Recall: True means the same thing as "always true" and False means the same thing as "there exists a counterexample".] No work is necessary for this problem. Unless stated otherwise, you may assume all random variables are discrete or continuous.

(a) __F__ Suppose X and Y are independent random variables. Then

$$\mathrm{var}(X-Y) = \mathrm{var}(X) + \mathrm{var}(Y).$$

(2)

(b) __T__ Suppose X is a discrete random variable with range $\{0,1,2,3,\ldots\}$ and PMF $p_X(x)$ and Y is a continuous random variable with PDF $f_Y(y)$. Then

$$\sum_{k=0}^{\infty} p_X(k) = \int_{-\infty}^{\infty} f_Y(y)dy. = 1$$

(2)

(c) __T__ Suppose A and B are independent and also suppose A and C are independent. Then A and $B \cup C$ are also independent. (2)

(d) __F__ If $\mathbf{P}(B) = 0$, then $B = \emptyset$. (2)

(e) __F__ Suppose A and B are independent and we are given a third event C. Then automatically A and B are conditionally independent with respect to C. (2)

(f) __T__ We want to distribute 20 distinguishable balls into 3 distinguishable buckets such that there are 10 balls in the first bucket, 7 balls in the second bucket and 3 balls in the third bucket. The total number of ways to do this is $\binom{20}{3,10,7}$. (2)

(g) __F__ Suppose X and Y are jointly continuous random variables. Then $f_{X|Y}(x|y) = f_X(x)f_{Y|X}(y|x)$. $/f_X(y)$ (2)

(h) __T__ Suppose X is a normal random variable with mean 10 and variance 4. Then $\mathbf{P}(X \leq 18)$ is equal to

$$\Phi\left(\frac{18-10}{4}\right) = \Phi(2),$$

where $\Phi(x)$ is the CDF of the standard normal random variable. (2)

(i) __T__ If $f_X : \mathbb{R} \to \mathbb{R}$ is the PDF of a continuous random variable X, then $0 \leq f_X(x) \leq 1$ for all $x \in \mathbb{R}$. (2)

2. For each of the random variables below, specify the following:

   (i) Defining parameters (e.g., for Bernoulli this is "$p \in [0,1]$")

   (ii) Range (e.g., for Bernoulli this is "$\{0,1\}$")

   (iii) Probability law: PMF or PDF or CDF,

   (iv) Expected value, and

   (v) Variance.

   The random variables are

   (a) Binomial. (5)

   (b) Geometric. (5)

   (c) Poisson. (5)

   (d) Continuous uniform. (5)

   (e) Gaussian/normal. (5)

   (f) Exponential. (5)

a. binomial: $n \in \mathbb{N}$, $0 \le p \le 1$ Range: $k \in \{0, 1, 2, \dots, n\}$

$f_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{in range} \\ 0 & \text{o/w} \end{cases}$

$E(X) = np$

$var(X) = np(1-p)$

b. Geometric: $0 \le p \le 1$

Range: $k \in \{1, 2, 3, \dots, \infty\}$

$f_X(k) = \begin{cases} (1-p)^{k-1} p & \text{in range} \\ 0 & \text{o/w} \end{cases}$

$E(X) = 1/p$

$var(X) = \frac{1-p}{p^2}$

c. Poisson: $\lambda > 0$, $\lambda \in \mathbb{R}$

Range: $k = \{0, 1, 2, \dots, \infty\}$

$f_X(k) = \begin{cases} e^{-\lambda} \frac{\lambda^k}{k!} & \text{in range} \\ 0 & \text{o/w} \end{cases}$

$E(X) = \lambda$

$Var(X) = \lambda$

d. continuous uniform: $a \in \mathbb{R}$, $b \in \mathbb{R}$, $a \le b$

Range: $x \in [a, b]$

$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{in range} \\ 0 & \text{o/w} \end{cases}$

$E(X) = \frac{b+a}{2}$

$Var(X) = \frac{(b-a)^2}{12}$

e. normal: $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}$, $\sigma \ge 0$

Range: $x \in \mathbb{R}$

$f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} & \text{in range} \\ 0 & \text{o/w} \end{cases}$

$E(X) = \mu$

$var(X) = \sigma^2$

f. exponential: $\lambda > 0$, $\lambda \in \mathbb{R}$

Range: $x \in (0, \infty]$, $x \in \mathbb{R}$

$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{in range} \\ 0 & \text{o/w} \end{cases}$

$E(X) = 1/\lambda$

$var(X) = 1/\lambda^2$

3. This question is about basic properties of probability laws.
   (a) State the axioms for Probability Laws (give names and mathematical definitions). (5)
   (b) Using the axioms, prove for all events A that $\mathbf{P}(A) \leq 1$. (5)

a. $P(\Omega)=1$ normalization

$P(A) \geq 0$ nonnegativity

If $A_1, A_2, \ldots, A_i$ are disjoint, $P(A_1 \cup A_2 \cup \ldots \cup A_i) = P(A_1) + P(A_2) + \ldots + P(A_i)$
countable additivity

b. $P(A) + P(A^c) = P(A \cup A^c) = P(\Omega)$ by countable additivity

$= 1$ by normalization

$P(A^c) \geq 0$ by nonnegativity

$\therefore P(A) \leq 1$ ∎

4. (a) A license plate consists of two letters and four digits (with possible repetition and in any order, e.g. $AB1234$ or $78C8C4$). How many different plates can be made up using the 26 letters A-Z of the English alphabet and the 10 digits 0-9? (5)

(b) On an $n \times n$ chessboard ($n \geq 2$), in how many ways can two castles of different colors be placed so that they cannot capture each other? In other words, in how many ways can two distinguishable objects be placed on an $n \times n$ board so that they are neither on the same column nor on the same row. (5)

a. $6! \cdot 26^2 \cdot 10^4$ ← 10^4 ways to choose digits

6! orderings ; 26^2 ways to choose letters

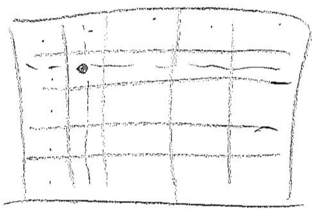
b. $n^2 (n-1)^2$

n^2 spots for first rook

can't go in same row ⇒ $(n-1)$ ×

" " column ⇒ $(n-1)$

$= (n-1)^2$ for second rook



$(b-a)(b^2+ab+a^2)$

$= b^3 + \cancel{ab^2} + \cancel{a^2b} - \cancel{ab^2} - a^3 - \cancel{a^2b}$

$Var(X) = \int_a^b \frac{x^2}{b-a}\,dx - \left(\frac{b+a}{2}\right)^2$

$= \frac{1}{b-a}\left(\frac{x^3}{3}\right)\Big|_a^b$

$= \frac{b^3-a^3}{3(b-a)} - \left(\frac{b+a}{2}\right)^2$

$= \frac{(b^2+ab+a^2)}{3} - \frac{b^2+2ab+a^2}{4}$

$= \frac{b^2+a^2-2ab}{12} = \frac{(b-a)^2}{12}$

5. Suppose X and Y are jointly continuous independent random variables such that X is Exponential λ (assume $\lambda > 1$) and Y is (continuous) Uniform on $[0, 1]$.

   (a) Calculate $\mathbf{E}[e^{X+Y}]$. (5)

   (b) Calculate $\mathrm{Var}(3X - 7Y + 3)$. (5)

a. $f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0 & o/w \end{cases}$

$f_Y(y) = \begin{cases} 1, & y \in (0,1) \\ 0 & o/w \end{cases}$

$E[e^{X+Y}] = E[e^X \cdot e^Y]$

$= E[e^X] E[e^Y]$

$= \int_0^\infty e^x \lambda e^{-\lambda x} dx \int_0^1 1 \cdot e^y dy$

$= \int_0^\infty \lambda e^{-(\lambda-1)x} dx \cdot (e^y)\big|_0^1$

$= \left(\frac{-\lambda e^{-(\lambda-1)x}}{\lambda-1}\right)\Big|_0^\infty \cdot (e-1)$

$= \boxed{\frac{\lambda(e-1)}{\lambda-1}}$

b. $\mathrm{var}(3X - 7Y + 3) = 9\,\mathrm{var}(X) + 49\,\mathrm{var}(Y)$

$= \frac{9}{\lambda^2} + 49 \cdot \frac{(1-0)^2}{12}$

$= \boxed{\frac{9}{\lambda^2} + \frac{49}{12}}$

6. Suppose X, Y are jointly continuous random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} Cxy & \text{if } x, y \geq 0 \text{ and } x + 2y \leq 2 \\ 0 & \text{otherwise,} \end{cases}$$

where C is some constant. To maximize partial credit, set up all integrals correctly.

(a) Determine C. (5)

(b) Compute the marginal PDF $f_X(x)$. (5)

(c) Compute the expected value $\mathbf{E}[X]$. (5)

a. $\int_0^1 \int_0^{2-2y} Cxy \, dx \, dy = 1$

$\int_0^1 C y \left(\frac{x^2}{2}\right)\Big|_0^{2-2y} dy = 1$

$\int_0^1 2C\, y(1-y)^2 \, dy = 1$

$\int_0^1 2C\,(y - 2y^2 + y^3) \, dy$

$= 2C\left(\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4}\right)\Big|_0^1$

$= 2C\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = 1$

$= \frac{C}{6} = 1$

$\boxed{C = 6}$

b. $f_X(x) = \int_0^{\frac{2-x}{2}} 6xy \, dy$

$= 6x\left(\frac{y^2}{2}\right)\Big|_0^{\frac{2-x}{2}}$

$= \frac{6x(2-x)^2}{8}$

$= \boxed{\begin{cases} \frac{3x(2-x)^2}{4}, & x \in [0,2] \\ 0 & \text{o/w} \end{cases}}$

c. $E(X)$

$= \int_0^2 x \cdot \frac{3x(2-x)^2}{4} \, dx$

$= \frac{3}{4}\int_0^2 4x^2 - 4x^3 + x^4 \, dx$

$= \frac{3}{4}\left(\frac{4x^3}{3} - x^4 + \frac{x^5}{5}\right)\Big|_0^2$

$= \frac{3}{4}\left(\frac{32}{3} - 16 + \frac{32}{5}\right)$

$= \frac{3}{4}\left(\frac{160}{15} - \frac{240}{15} + \frac{96}{15}\right)$

$= \frac{3}{4}\left(\frac{16}{15}\right)$

$= \boxed{\frac{4}{5}}$

Check: $\int_0^2 \int_0^{\frac{2-x}{2}} 6xy \, dy \, dx$

$= \int_0^2 6x \frac{(2-x)^2}{8} \, dx$

$= \frac{3}{4} \quad 4x - 4x^2 + x^3$

$= \frac{3}{4}\left(2x^2 - \frac{4x^3}{3} + \frac{x^4}{4}\right)\Big|_0^2$

$= \frac{3}{4}\left(8 - \frac{32}{3} + \frac{16}{4}\right) = 1$ ✓

7. A defective coin minting machine produces coins whose probability of heads is a random variable P with PDF

$$f_P(p) = \begin{cases} \frac{e^p}{e-1} & \text{if } p \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

A coin produced by this machine is selected and tossed repeatedly, with successive tosses assumed independent.

(a) Let A be the event that the first toss is heads. Compute $\mathbf{P}(A)$. (5)

(b) Let B be the event that the second toss is heads. Compute $\mathbf{P}(B|A)$. (5)

(c) Are A and B independent? Justify your answer. (2)

a. $P(A) = \int_0^1 P(A | P = p) f_P(p)\, dp$

$= \int_0^1 p \frac{e^p}{e-1}\, dp$

$= \left(\frac{pe^p}{e-1} - \frac{e^p}{e-1} \right) \Big|_0^1$

$= \frac{1 \cdot e^1}{e-1} - \frac{e^1}{e-1} - \left(0 - \frac{e^0}{e-1} \right)$

$= \boxed{\frac{1}{e-1}}$

b. $P(B|A) = \int_0^1 P(B | P = p, A) f_P(p)\, dp$

$= \int_0^1 p^2 \frac{e^p}{e-1}\, dp$ $\quad u = p^2 \quad du = 2p \quad v = \frac{e^p}{e-1} \quad dv = \frac{e^p}{e-1}$

$= \left(p^2 \frac{e^p}{e-1} \right) \Big|_0^1 - \int_0^1 2p \frac{e^p}{e-1}\, dp$ ↘ solved in part (a)

$= \frac{e}{e-1} - \frac{2}{e-1}$

$= \boxed{\frac{e-2}{e-1}}$

c. no, $P(B) = \frac{1}{e-1} \neq P(B|A) = \frac{e-2}{e-1}$

8. (a) State *Markov's Inequality.* (5)

(b) State *Chebyshev's Inequality.* (5)

(c) Suppose X is a random variable such that $\mathbf{E}[X] = 1$ and $\text{Var}(X) = 0.04$. Give a lower bound on the probability that $0.5 < X < 1.5$. Your lower bound must be greater than 0 and you must justify your answer. (5)

a. $P(X \geq a) \leq \frac{E(x)}{a}$

b. $P(|X - a| \geq \mu) \leq \frac{Var(x)}{a^2}$

c. $E(x) = 1 \quad Var(X) = 0.04$

$P(0.5 < X < 1.5)$

$= 1 - P(X \leq 0.5 \;\&\; X \geq 1.5)$

$= 1 - P(|X - 0.5| \geq 1)$, $P(|X - 0.5| \geq \mu) \leq \frac{Var(x)}{0.5^2}$ by Chebyshev's inequality

$\geq 1 - \frac{0.04}{0.5^2}$

$\geq 1 - \frac{4}{25}$

$\geq \boxed{0.84}$