1. (40 points) Let $X$ be a random variable with PMF

$$p_X(x) = \begin{cases} \frac{x^2}{a} & \text{if } x = -2,-1,0,1,2,3,4 \\ 0 & \text{otherwise} \end{cases}$$

40

(a) (10 points) Find $a$ and $\mathbf{E}[X]$.

(b) (10 points) What is the PMF of the random variable $Z = (X - \mathbf{E}[X])^2$ ?

(c) (10 points) Using the result from part (b), find the variance of $X$.

(d) (10 points) Find the variance of $X$ using the formula $\text{var}(X) = \sum_x (X - \mathbf{E}[X])^2 p_X(x)$.

a)

$1 = \frac{(-2)^2}{a} + \frac{(-1)^2}{a} + \frac{0^2}{a} + \frac{1^2}{a} + \frac{2^2}{a} + \frac{3^2}{a} + \frac{4^2}{a}$

$1 = \frac{4}{a} + \frac{1}{a} + \frac{1}{a} + \frac{4}{a} + \frac{9}{a} + \frac{16}{a}$

$1 = \frac{35}{a}$

$a = 35$

$E[X] = \sum_{i=-2}^{4} x_i\, p_X(x_i) = (-2)\frac{(-2)^2}{35} + (-1)\cdot\frac{(-1)^2}{35} + 0 + 1\cdot\frac{1^2}{35} + 2\cdot\frac{2^2}{35} + 3\cdot\frac{3^2}{35} + 4\cdot\frac{4^2}{35}$

$= \frac{-8}{35} + \frac{-1}{35} + \frac{1}{35} + \frac{8}{35} + \frac{27}{35} + \frac{64}{35}$

$= \frac{91}{35}$ #

b) when $x=-2$ $\quad z = (-2-\frac{91}{35})^2 = \frac{529}{25}$

when $x=-1$ $\quad z = (-1-\frac{91}{35})^2 = \frac{324}{25}$

when $x=0$ $\quad z = (0-\frac{91}{35})^2 = \frac{91^2}{35^2}$

when $x=1$ $\quad z = (1-\frac{91}{35})^2 = \frac{64}{25}$

when $x=2$ $\quad z = (2-\frac{91}{35})^2 = \frac{9}{25}$

when $x=3$ $\quad z = (3-\frac{91}{35})^2 = \frac{4}{25}$

when $x=4$ $\quad z = (4-\frac{91}{35})^2 = \frac{49}{25}$

$a \cdot 4^2$

$\times 4^2 = 1.96$

$$p_Z(z) = \begin{cases} \frac{4}{35} & \text{if } z = \frac{529}{25} \\ \frac{1}{35} & \text{if } z = \frac{324}{25} \\ \frac{1}{35} & \text{if } z = \frac{64}{25} \\ \frac{4}{35} & \text{if } z = \frac{9}{25} \\ \frac{9}{35} & \text{if } z = \frac{4}{25} \\ \frac{16}{35} & \text{if } z = \frac{49}{25} \\ 0 & \text{otherwise} \end{cases}$$

c) $Var(X) = E[X-E[X]]^2$

$E[Z] = E[X-E[X]]^2 = Var(X)$

$E[Z] = \frac{4}{35}\times\frac{529}{25} + \frac{1}{35}\times\frac{324}{25} + \frac{1}{35}\times\frac{64}{25} + \frac{4}{35}\times\frac{9}{25} + \frac{9}{35}\times\frac{4}{25} + \frac{16}{35}\times\frac{49}{25} = \frac{96}{25}$ #

$\therefore Var(X) = \frac{96}{25}$

d). $Var(X) = \sum_x (X - E[X])^2 p_X(x)$

$= E[X^2] - (E[X])^2$

$E[X^2] = \sum_x x^2 p_X(x) = (-2)^2 \frac{(-2)^2}{35} + (-1)^2 \frac{(-1)^2}{35} + 0 + (1)^2 \frac{1^2}{35} + 2^2 \frac{2^2}{35} + 3^2 \frac{3^2}{35} + 4^2 \frac{4^2}{35}$

$= \frac{53}{5}.$

$Var(X) = E[X^2] - (E[X])^2$

$= \frac{53}{5} - \left(\frac{91}{35}\right)^2 = \frac{96}{25}.$ which is the same as the answer for part c)

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60

2. (60 points) A coin that has probability of heads equal to $p$ is tossed successively and independently until a head comes twice in a row. Denote

$$X = \text{the number of tosses until the game is over}$$

(a) (30 points) Find $\mathbf{E}[X]$;

(b) (30 points) Find $\text{var}(X)$.

a). let $H_i$ represent head and $T_i$ represent tail at $i^{th}$ toss.

If $\Omega$ is all the outcome of a single toss, then $H_1 + T_1 = \Omega$. (partition)
the first toss is either a head or a tail.
Similarly $H_i + T_i = \Omega$ for all $i \in \mathbb{Z}^+$

$$E[X] = E[X \mid H_1] P(H_1) + E[X \mid T_1] P(T_1)$$

$$= pE[X \mid H_1] + (1-p) E[X \mid T_1]$$

$$= pE[X \mid H_1 \cap H_2] P(H_2) + pE[X \mid H_1 \cap T_2] P(T_2) + (1-p) E[X \mid T_1]$$

$$= p^2 E[X \mid H_1 \wedge H_2] + p(1-p) E[X \mid H_2 \wedge T_2] + (1-p)(1 + E[X])$$

$$= p^2 \times 2 + p(1-p)(E[X] + 2) + (1-p)(1 + E[X])$$

Since if the first toss is tail, we can treat the second toss as the starting point and add the expectation by 1, similarly for if $1^{st}$ is head and second is tail.

we can rewrite it as

$$E[X] = 2p^2 + 2p(1-p) + p(1-p)E[X] + (1-p) + (1-p)E[X]$$

$$\cancel{E[X]} = \cancel{2p^2} + 2p - \cancel{2p^2} + \cancel{pE[X]} - p^2 E[X] + 1 - p + \cancel{E[X]} - \cancel{pE[X]}$$

$$0 = 1 + p - p^2 E[X]$$

$$p^2 E[X] = 1 + p$$

$$E[X] = \frac{1+p}{p^2} \quad \#$$

b) $Var(X) = E[X^2] - [E[X]]^2$

1st consider, $E[X^2] = E[X^2|H_1]P(H_1) + E[X^2|T_1]P(T_1)$

$$= pE[X^2|H_1] + (1-p)(E[(X+1)^2])$$

$$= pE[X^2|H_1 \cap H_2]P(H_2) + pE[X^2|H_1 \cap T_2]P(T_2) + (1-p)(E[X^2+2X+1])$$

$$= p^2E[X^2|H_1 \cap H_2] + p(1-p)E[X^2|H_1 \cap T_2] + (1-p)(E[X^2]+2E[X]+1)$$

$$= p^2 \cdot 2^2 + p(1-p)E[(X+2)^2] + (1-p)E[X^2] + 2(1-p)E[X] + (1-p)$$

$$= 4p^2 + (p-p^2)E[X^2+4X+4] + (1-p)E[X^2] + 2(1-p)E[X] + (1-p)$$

$$= 4p^2 + (p-p^2)E[X^2] + 4(p-p^2)E[X] + 4(p-p^2) + (1-p)E[X^2] + 2(1-p)E[X] + 1 - p$$

$$= E[X^2](p-p^2+1-p) + E[X](4p-4p^2+2-2p) + 4p^2 + 4p - 4p^2 + 1 - p$$

$$E[X^2] = E[X^2](1-p^2) + E[X](-4p^2+2p+2) + 3p+1$$

$$p^2E[X^2] = \frac{1+p}{p^2}(-4p^2+2p+2) + 3p+1$$

$$E[X^2] = \frac{1+p}{p^4}(2p-4p^2+2) + \frac{3p+1}{p^2}$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$= \frac{1+p}{p^4}(2p-4p^2+2) + \frac{3p+1}{p^2} - \frac{(1+p)^2}{p^4}$$

$$= \frac{\cancel{2p} - \cancel{4p^2} + 2 + \cancel{2p^2} - 4p^3 + \cancel{2p} - 1 - \cancel{p^2} - \cancel{2p}}{p^4} + \frac{3p^3+p^2}{p^4}$$

$$= \frac{-4p^3-3p^2+2p+1}{p^4} + \frac{3p^3+p^2}{p^4}$$

$$= \frac{-p^3-2p^2+2p+1}{p^4}$$

#

$$\therefore Var(X) = \frac{-p^3-2p^2+2p+1}{p^4}$$