

1. (60 points) Each of  $k$  jars (labeled jar 1, jar 2,  $\dots$ , jar  $k$ ) contains  $m$  white and  $n$  black balls. We perform the following exchanges:

Step 1 We pick simultaneously and at random a ball from jar 1 and a ball from jar 2. Then switch these two balls.

Step 2 We pick simultaneously and at random a ball from jar 2 and a ball from jar 3. Then switch these two balls.

$\dots$

Step  $k-1$  We pick simultaneously and at random a ball from jar  $k-1$  and a ball from jar  $k$ . Then switch these two balls.

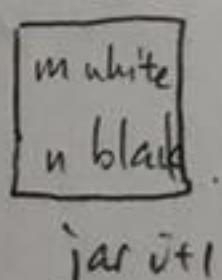
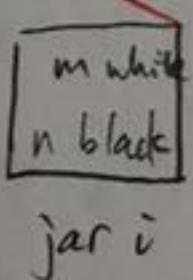
Step  $k$  We pick a ball at random from jar  $k$ .

Please answer the following questions:

(a) (40 points) Develop a recursive formula that allows the convenient computation of the probability that the ball picked at Step  $k$  is white.

(b) (20 points) Let us take  $m = 1$ ,  $n = 2$  and  $k = 3$ . Calculate the probability that the ball picked at Step 3 is white.

a)  
and b)



The total number of balls in a jar is always  $m+n$ .  
where  $1 \leq i \leq k-1$   $i \in \mathbb{Z}$

Let's denote  $(m, n)_i$  be the case where there are  $m$  white balls and  $n$  black balls in jar  $i$ .

Possible outcome after 1 single exchange.

① exchange  $(m, n)_i, (m, n)_{i+1}$   
the same color

② exchange different colors

OR

Denote  $A_k$  be the event when a white ball is picked at step  $k$ .

Then

The number of black balls in jar  $i$

$$\text{when } k=1 \quad P(A_1) = \frac{m}{m+n}$$

$$\begin{aligned} \text{when } k=2 \quad P(A_2) &= \left[ \left( \frac{m}{m+n} \right)^2 + \left( \frac{n}{m+n} \right)^2 \right] \frac{m}{m+n} + \frac{mn}{(m+n)^2} \frac{m-1}{m+n} + \frac{mn}{(m+n)^2} \frac{m+1}{m+n} \\ &= \frac{m}{m+n} \left[ \left( \frac{m}{m+n} \right)^2 + \left( \frac{n}{m+n} \right)^2 + \frac{2mn}{(m+n)^2} \right] \\ &= \frac{m}{m+n} \left[ \frac{m^2}{m+n} + \frac{n^2}{m+n} \right]^{\frac{1}{2}} = \frac{m}{m+n} \end{aligned}$$

when  $k=3$ .

$$\begin{aligned} P(A_3) &= \left[ \left( \frac{m}{m+n} \right)^2 + \left( \frac{n}{m+n} \right)^2 \right] \frac{m}{m+n} + \frac{mn}{(m+n)^2} \frac{m-1}{m+n} + \frac{mn}{(m+n)^2} \frac{m+1}{m+n} \\ &= \frac{m}{m+n} \# \end{aligned}$$

$\Rightarrow$  turn to next page

$$\text{By Induction } P(A_k) = \frac{m}{m+n} \quad \#$$

Switch between  $k_1^{\text{th}}$  and  $k^{\text{th}}$  jar.

$P_k$  = "probability that white ball picked from  $k^{\text{th}}$  jar."

$$P_k = P(W, W) P(W | (W, W)) + P(W, B) P(W | (W, B)) \\ + P(B, W) P(W | (B, W)) + P(B, B) P(W | (B, B)).$$

$$P(W, W) = P_{k-1} \frac{m}{m+n}$$

$$P(W | (W, W)) = \frac{m}{m+n}$$

$$P(W, B) = P_{k-1} \frac{n}{m+n}$$

$$P(W | (W, B)) = \frac{m+1}{m+n}$$

$$P(B, W) = (1 - P_{k-1}) \frac{m}{m+n}$$

$$P(W | (B, W)) = \frac{m-1}{m+n}$$

$$P(B, B) = (1 - P_{k-1}) \frac{n}{m+n}$$

$$P(W | (B, B)) = \frac{n}{m+n}$$

2. (40 points) Suppose we have a biased coin and a fair 4-sided die. Let us perform the following two-step experiment:

**Step 1:** Toss the coin once.

**Step 2:** If the coin toss results a head, we roll the die once; if the coin toss results a tail, we roll the die twice.

Let us consider the following two events:

$$A = \{\text{the maximum of the die roll(s)} = 4\}$$

and

$$B = \{\text{the total sum of the die roll(s)} = 4\}$$

Question: Assume that the event  $A$  and  $B$  are independent, what is the probability that the result of the Step 1 is a head?

Let  $C = \{\text{coin toss results a head}\}$ .

we know that  $P(A \cap B) = P(A)P(B)$ .

$$P(A) = P(C)P(A|C) + P(C^c)P(A|C^c)$$

$$P(A|C) = P(\{4\}) = \frac{1}{4}$$

$$P(A|C^c) = P(\{(1,4), (2,4), (3,4), (4,4), (4,1), (4,2), (4,3)\})$$

$$= \frac{7}{16}$$

$$\therefore P(A) = \frac{1}{4}P(C) + \frac{7}{16}(1 - P(C)) = \frac{1}{4}P(C) + \frac{7}{16} - \frac{7}{16}P(C) = \underline{\underline{\frac{7}{16} - \frac{3}{16}P(C)}} \quad \checkmark$$

$$P(B) = P(C)P(B|C) + P(C^c)P(B|C^c)$$

$$P(B|C) = P(\{4\}) = \frac{1}{4}$$

$$P(B|C^c) = P(\{(1,3), (2,2), (3,1)\}) = \frac{3}{16}$$

$$\therefore P(B) = \frac{1}{4}P(C) + \frac{3}{16}(1 - P(C)) = \frac{1}{4}P(C) + \frac{3}{16} - \frac{3}{16}P(C) = \underline{\underline{\frac{3}{16} + \frac{1}{16}P(C)}} \quad \checkmark$$

$$P(A \cap B) = P(C)P(A \cap B|C) + P(C^c)P(A \cap B|C^c)$$

$$P(A \cap B|C) = P(\{4\}) = \frac{1}{4}$$

$$P(A \cap B|C^c) = P(\emptyset) = 0$$

$$\therefore P(A \cap B) = \frac{1}{4}P(C) + 0 \times (1 - P(C)) = \underline{\underline{\frac{1}{4}P(C)}} \quad \checkmark$$

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$$\because P(A \cap B) = P(A)P(B).$$

$$\frac{1}{4}P(C) = \left(\frac{7}{16} - \frac{3}{16}P(C)\right) \cdot \left(\frac{3}{16} + \frac{1}{16}P(C)\right) \quad \checkmark$$

$$\text{Let } P(C) = k$$

$$\frac{1}{4}k = \left(\frac{7}{16} - \frac{3}{16}k\right)\left(\frac{3}{16} + \frac{1}{16}k\right).$$

$$64k = (7 - 3k)(3 + k).$$

$$64k = 21 + 7k - 9k - 3k^2.$$

$$3k^2 + 66k - 21 = 0.$$

$$k^2 + 22k - 7 = 0.$$

$$k = \frac{-22 \pm \sqrt{22^2 + 28}}{2}$$

$$\begin{array}{r} 3 & 7 \\ 1 & 3 \\ \hline 4 & 128 \\ & 4 \cancel{3}2 \\ & 4 \cancel{1}8 \\ & \hline & 2 \end{array} \quad \begin{array}{r} 1.41 \\ -3.18 \\ \hline 28 \end{array}$$

$$\text{Since, } P(C) \geq 0$$

$$\text{we choose } k = \frac{\sqrt{22^2 + 28} - 22}{2} = -11 + \sqrt{11^2 + 7} = 8\sqrt{2} - 11. \quad \checkmark$$

$$\therefore P(C) = 8\sqrt{2} - 11 \quad \#$$