Fall 2017: Math 170 ...

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1. (60 points) Each of k jars (labeled jar 1, jar 2, $\cdots$, jar k) contains m white and n black balls. We perform the following exchanges:

Step 1 We pick simultaneously and at random a ball from jar 1 and a ball from jar 2. Then switch these two balls.

Step 2 We pick simultaneously and at random a ball from jar 2 and a ball from jar 3. Then switch these two balls.

...

Step $k-1$ We pick simultaneously and at random a ball from jar $k-1$ and a ball from jar k. Then switch these two balls.

Step k We pick a ball at random from jar k.

Please answer the following questions:

(a) (40 points) Develop a **recursive formula** that allows the convenient computation of the probability that the ball picked at Step k is white.

(b) (20 points) Let us take $m = 1$, $n = 2$ and $k = 3$. Calculate the probability that the ball picked at Step 3 is white.

a) and b)

m white, n black	m white, n black
jar i	jar $i+1$

The total number of balls in a jar is always $m+n$, where $1 \le i \le k-1$, $i \in \mathbb{Z}$

Let's denote $(m, n)_i$ be the case where there are m white balls and n black balls in jar i.

possible outcome after 1 single exchange.

① exchange the same color $(m, n)_i$, $(m, n)_{i+1}$

② exchange different colors $(m-1, n+1)_i$, $(m+1, n-1)_{i+1}$ OR $(m+1, n-1)_i$, $(m-1, n+1)_{i+1}$

Denote A_k be the event when a white ball is picked at step k.

when $k=1$ $\quad P(A_1) = \frac{m}{m+n}$

when $k=2$

$$P(A_2) = \left[\left(\frac{m}{m+n}\right)^2 + \left(\frac{n}{m+n}\right)^2\right]\frac{m}{m+n} + \frac{mn}{(m+n)^2}\,\frac{m-1}{m+n} + \frac{mn}{(m+n)^2}\,\frac{m+1}{m+n}$$

$$= \frac{m}{m+n}\left[\left(\frac{m}{m+n}\right)^2 + \left(\frac{n^2}{m+n}\right)^2 + \frac{2mn}{(m+n)^2}\right]$$

$$= \frac{m}{m+n}\left(\frac{m}{m+n} + \frac{n}{m+n}\right)^2 = \frac{m}{m+n}$$

when $k=3$.

$$\underline{P(A_3)} = \left[\left(\frac{m}{m+n}\right)^2 + \left(\frac{n}{m+n}\right)^2\right]\frac{m}{m+n} + \frac{mn}{(m+n)^2}\,\frac{m-1}{m+n} + \frac{mn}{(m+n)^2}\,\frac{m+1}{m+n}$$

$$= \underline{\frac{m}{m+n}} \quad \#$$

$\Rightarrow$ turn to next pa

[By Induction $P(A_k) = \frac{m}{m+n}$]

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Switch between $k-1^{th}$ and k^{th} jar.

P_k = "probability that white ball picked from k^{th} jar".

$$P_k = P(W,W)\,P(W\,|\,(W,W)) + P(W,B)\,P(W\,|\,(W,B)) + P(B,W)\,P(W\,|\,(B,W)) + P(B,B)\,P(W\,|\,(B,B)).$$

$P(W,W) = P_{k-1}\frac{m}{m+n}$ $\qquad$ $P(W\,|\,(W,W)) = \frac{m}{m+n}$

$P(W,B) = P_{k-1}\frac{n}{m+n}$. $\qquad$ $P(W\,|\,(W,B)) = \frac{m+1}{m+n}$

$P(B,W) = (1-P_{k-1})\frac{m}{m+n}$ $\qquad$ $P(W\,|\,(B,W)) = \frac{m-1}{m+n}$.

$P(B,B) = (1-P_{k-1})\frac{n}{m+n}$. $\qquad$ $P(W\,|\,(B,B)) = \frac{m}{m+n}$

2. (40 points) Suppose we have a **biased coin** and a **fair 4-sided die**. Let us perform the following two-step experiment:

Step 1: Toss the coin once.

Step 2: If the coin toss results a head, we roll the die once; if the coin toss results a tail, we roll the die twice.

Let us consider the following two events:

$$A = \{\text{the maximum of the die roll(s)} = 4\}$$

and

$$B = \{\text{the total sum of the die roll(s)} = 4\}$$

Question: Assume that the event A and B are **independent**, what is the probability that the result of the **Step 1** is a head?

Let $C = \{$coin toss results a head$\}$.

we know that $P(A \cap B) = P(A)P(B)$.

$P(A) = P(C)P(A|C) + P(C^c)P(A|C^c)$.

$P(A|C) = P(\{4\}) = \frac{1}{4}$.

$P(A|C^c) = P(\{(1,4),(2,4),(3,4),(4,4),(4,1),(4,2),(4,3)\})$

$= \frac{7}{16}$.

when toss a head, we have the possible set of die roll $\{1,2,3,4\}$ with each has possibility of $\frac{1}{4}$
when toss a tail, we have the possible set of die rolls $\{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}$. each outcome has possibility $\frac{1}{16}$.

$\therefore P(A) = \frac{1}{4}P(C) + \frac{7}{16}(1-P(C)) = \frac{1}{4}P(C) + \frac{7}{16} - \frac{7}{16}P(C) = \underline{\frac{7}{16} - \frac{3}{16}P(C)}$ ✓

$P(B) = P(C)P(B|C) + P(C^c)P(B|C^c)$

$P(B|C) = P(\{4\}) = \frac{1}{4}$.

$P(B|C^c) = P(\{(1,3),(2,2),(3,1)\}) = \frac{3}{16}$.

$\therefore P(B) = \frac{1}{4}P(C) + \frac{3}{16}(1-P(C)) = \frac{1}{4}P(C) + \frac{3}{16} - \frac{3}{16}P(C) = \underline{\frac{3}{16} + \frac{1}{16}P(C)}$ ✓

$P(A \cap B) = P(C)P(A \cap B|C) + P(C^c)P(A \cap B|C^c)$.

$P(A \cap B|C) = P(\{4\}) = \frac{1}{4}$

$P(A \cap B|C^c) = P(\phi) = 0$.

$\therefore P(A \cap B) = \frac{1}{4}P(C) + 0 \times (1-P(C)) = \underline{\frac{1}{4}P(C)}$ ✓

⇒ next page.

$\because \quad P(A \cap B) = P(A)\,P(B).$

$$\frac{1}{4}P(C) = \left(\frac{7}{16} - \frac{3}{16}P(C)\right)\cdot\left(\frac{3}{16} + \frac{1}{16}P(C)\right)$$

Let $P(C) = k$

$$\frac{1}{4}k = \left(\frac{7}{16} - \frac{3}{16}k\right)\left(\frac{3}{16} + \frac{1}{16}k\right).$$

$$64k = (7-3k)(3+k).$$

$$64k = 21 + 7k - 9k - 3k^2.$$

$$3k^2 + 66k - 21 = 0.$$

$$k^2 + 22k - 7 = 0.$$

$$k = \frac{-22 \pm \sqrt{22^2 + 28}}{2}$$

Since $P(C) \geq 0$

we choose $k = \dfrac{\sqrt{22^2+28} - 22}{2} = -11 + \sqrt{11^2+7} = 8\sqrt{2} - 11.$

$\therefore \quad P(C) = 8\sqrt{2} - 11$

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