Final exam, Math 170A, Spring 2020
Instructor: Liza Rebrova

Printed name: ______________________________

Signed name: ______________________________

Student ID number: ______________________________

Instructions:

- On the first page of the work, everyone must state in writing *"I assert, on my honor, that I have not received assistance of any kind from any other person while working on the final and that I have not used any non-permitted materials or technologies during the period of this evaluation.* " Please recall that UCLA has Student Conduct Code (it can be found at www.deanofstudents.ucla.edu ; see, in particular, Section 102.01 on academic dishonesty).
- Any collaboration (personal or on the internet) is not allowed.
- Books, arithmetic calculators, googling are allowed.
- The correct answer for any problem is not sufficient for full credit, you should carefully explain each solution (referring only to the official textbook or class materials).
- Read problems very carefully. If you have any questions please email us at `rebrova@math.ucla.edu` and `bringmann@math.ucla.edu`.
- Keep an eye on CCLE announcements. If I get many similar questions, I will make a clarifying announcement for everyone.
- **Your work should be submitted at Gradescope by 8am on Thursday (June 11th). Please check the quality of your photos before submission! Please separate the parts belonging to different problems.**
- You have 8 problems, 90 points total. Good luck!

1. Provide all necessary computations/explanations for your answers.

   (a) (5 points) Let X and Y be independent random variables such that $\mathbb{E}[X] = \mathbb{E}[Y] = 1$ and $\text{var}[X] = \text{var}[Y] = 2$. Let $Z = X - Y$. Compute $\mathbb{E}[ZX]$.

   (b) (5 points) Let F_X be the distribution function of a random variable X and let $Y = -X$. Prove that Y is also a random variable.

   (c) (5 points) For random variables X and Y from the previous part, express the distribution function of Y in terms of F_X (namely, for any $t \in \mathbb{R}$, what is $F_Y(t) =$?)

   (d) (5 points) On each trial two dice are rolled at the same time and the sum of the dice is recorded. If 20 independent trials are conducted, what is the probability a sum of 3 was recorded exactly 5 times?

   (e) (5 points) Let $X, Y \sim \mathcal{N}(\mu, \sigma^2)$ be independent random variables, and find $\mathbb{P}[X > Y]$.

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2. (10 points) Let Y be a random variable uniformly distributed on $\{0, 1, \ldots, 10\}$ (the set of 11 integers) and Z be a uniform random variable on $[0, 10]$ (a segment from 0 to 10). Let

$$X_1 = \max(5, \min(Y, 7)) \text{ and } X_2 = \max(5, \min(Z, 7)).$$

   (a) (6 points) Find distribution functions of X_1 and X_2.

   (b) (4 points) For each of X_1 and X_2, state whether it is discrete, continuous, or neither. Justify your answers.

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3. (10 points) Let A be the subset of the plane defined as the intersection of the first quadrant and the annulus between the circles of radii 1 and 2 centered at the origin. In other words

$$A = \{(x, y) | x \geq 0, y \geq 0, 1 \leq x^2 + y^2 \leq 4\}.$$

Let (X, Y) be a uniformly chosen point in the region A. Find the marginal density function and the expected value of the first coordinate X.

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4. (10 points) A box contains n balls, $n \geq 3$, exactly one of which is red and one of which is yellow. We draw a ball uniformly at random, observe the color, return it back to the

box, and repeat this indefinitely. Let X be the number of the draw on which we first time obtain the yellow ball, and let Y be the number of the draw on which we first time obtain the red ball.

(a) (5 points) Compute $\mathbb{E}[X+Y]$. Are X and Y independent? Justify your answer.

(b) (5 points) Find joint probability mass function of (X, Y).

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5. (10 points) There are 20 Statistics students and 20 Mathematics students. They are randomly split into 20 study pairs, with 2 students per study pair. All such pairings are equally probable. Find expected number of pairs consisting of 1 Statistics student and 1 Mathematics student.

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6. (10 points) Two percent of LA citizens are wizards, the other are muggles. Owls can tell if a person they meet is a wizard or a muggle, but young owlets have not polished this skill to perfection yet. Namely, they can correctly detect a wizard with probability 90% (if the person was actually a wizard, in 90% of the cases they say so), and additionally with 5% chance they call a muggle a wizard. Two young owlets fly together and see a new person.

   (a) (5 points) What is the probability that both owlets disagree with each other?

   (b) (5 points) If both owlets say that the person is a wizard, what is the probability that he is actually a wizard?

7. Consider the following joint density function

$$f(x,y) = \begin{cases} \frac{2}{4+\pi}(y+\sin(x)) & \text{if } 0 < x < \pi \text{ and } 0 < y < 1 \\ 0 & \text{else} \end{cases}$$

You do not need to prove that f is a joint density function. You may take that for granted.

(a) (5 points) Find the conditional density of Y given $X = x$ for all values of $x \in \mathbb{R}$ where it is defined.

(b) (5 points) Are X and Y independent? Are they uncorrelated?

8. (5 points) Events $A_1, \ldots, A_n, \ldots$ are independent and have probability p each. Prove that

$$\mathbb{P}(\cap_{i=1}^{\infty} A_i) = \begin{cases} 1 & \text{if } p = 1 \\ 0 & \text{otherwise} \end{cases}$$

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[redacted]

TOTAL POINTS

86 / 90

QUESTION 1

1 1a **3 / 5**

+ **5 pts** Correct (=2)

✓ + **3 pts** **Correct approach, arithmetic error**

+ **1 pts** Wrong approach; E[X^2] is not E[X]^2

+ **0 pts** Missing

1 This should be a +

QUESTION 2

2 1b **5 / 5**

✓ + **5 pts** **Correct**

+ **4 pts** Mostly Correct, but inaccurate switch from <= to <

+ **4 pts** Mostly correct, but contains a minor mistake

+ **3 pts** Only checked {Y=y}, which is only enough for discrete rv's. Need {Y<=y}.

+ **2 pts** Wrote out set {Y<=y}, but wrong or no further details

+ **0 pts** Not addressing the question

+ **0 pts** Missing

QUESTION 3

3 1c **5 / 5**

✓ + **5 pts** **Correct**

+ **4 pts** Avoided problem with strict inequality by inserting P(X=-t), which is not in terms of F. Else correct.

+ **3 pts** Correct for continuous random variables; but limit missing for general case. Result of a mistake with strict vs non-strict inequality

+ **2 pts** F_X instead of 1-F_X

+ **0 pts** (Seriously) wrong answer. For instance, negative, not monotone,...

+ **0 pts** Missing

QUESTION 4

4 1d **5 / 5**

✓ + **5 pts** **Correct**

+ **3 pts** Factors correct, but binomial coefficient missing.

+ **3 pts** Binomial coefficient there, but one of the factors (1/18)^5 or (17/18)^(15) missing.

+ **1 pts** Got the 1/18 for a single roll, but combined probability missing at least two things.

+ **0 pts** Missing

QUESTION 5

5 1e **5 / 5**

✓ + **5 pts** **Correct (=1/2)**

+ **3 pts** The correct result, but even being nice not enough details. However, integral is used, so X=Y is addressed.

+ **3 pts** Almost corrrect, except for incorrect treatment of P(X=Y) or ignoring this issue.

+ **3 pts** Symmetry idea, but several incorrect statement (Working with P(X=x), for instance).

+ **1 pts** (Long) calculation, but no success

+ **0 pts** Wrong approach

+ **0 pts** Missing

2 Great! There is a quicker way that relies on P(X>Y)=P(X<Y) (by symmetry)

QUESTION 6

6 2a **6 / 6**

✓ - **0 pts** **Correct**

QUESTION 7

7 2b **4 / 4**

✓ - **0 pts** **Correct**

QUESTION 8

8 3 **10 / 10**

- ✓ **+ 10 pts** Correct
- **+ 6 pts** Marginal
- **+ 3 pts** Marginal, one of the regions 0<= x <=1 or 1<= x <=2 completely missing
- **+ 4 pts** Expectation
- **+ 3 pts** Expectation; but mistake. The result seems sensible, however, so could not be easily detect.
- **+ 2 pts** Expectation; but mistake. The mistake can be found by just thinking about the picture.
- **+ 3 pts** Expectation; but with wrong marginal. The mistake could be seen from a quick check, e.g., too close to 2.
- **+ 2 pts** Only abstract definitions in both cases.
- **+ 1 pts** Wrong definition of uniform distribution; other substantial errors
- **+ 0 pts** Missing

QUESTION 9

9 4a **5 / 5**

- ✓ **- 0 pts** Correct

QUESTION 10

10 4b **5 / 5**

- ✓ **- 0 pts** Correct

QUESTION 11

11 5 **10 / 10**

- ✓ **- 0 pts** Correct

QUESTION 12

12 6a **5 / 5**

- ✓ **+ 5 pts** **Correct (approx. 0.0967)**
- **+ 4 pts** Computational error, but otherwise correct
- **+ 3 pts** Correct conditional probabilities, but missing P(muggle) and P(wizard)
- **+ 3 pts** Correct approach, but took different probabilities than in problem statement.
- **+ 1 pts** Incorrect answer, major conceptual error
- **+ 1 pts** Incorrect answer, due to incorrect use of independence
- **+ 0 pts** Missing

QUESTION 13

13 6b **5 / 5**

- ✓ **+ 5 pts** **Correct (324/373)**
- **+ 5 pts** Correct approach (error propagating from (a) leads to wrong result)
- **+ 3 pts** Correct use of bayes
- **+ 2 pts** Incorrect use of bayes or total probability
- **+ 0 pts** Major conceptual error

QUESTION 14

14 7a **5 / 5**

- ✓ **- 0 pts** Correct

QUESTION 15

15 7b **5 / 5**

- ✓ **- 0 pts** Correct

QUESTION 16

16 8 **3 / 5**

- ✓ **- 2 pts** **Why is it possible to take the limit for probabilities/probability of the infinite intersection is the infinite product?**

QUESTION 1

a| $\mathbb{E}[ZX] = \mathbb{E}[(X-Y)X]$ since $Z = X-Y$

$= \mathbb{E}[X^2 - XY]$

$= \mathbb{E}[X^2] - \mathbb{E}[XY]$ (by Linearity of Expectations)

Now $\mathrm{Var}X = \mathbb{E}X^2 - (\mathbb{E}X)^2$

Given, $\mathrm{Var}X = 2$, $\mathbb{E}X = 1$

we get, $\mathbb{E}X^2 = \mathrm{Var}X - (\mathbb{E}X)^2 = 2 - 1 = 1$

Since X, Y, independent, $\mathbb{E}[XY] = \mathbb{E}X \cdot \mathbb{E}Y$

$\Rightarrow \mathbb{E}[XY] = 1 \cdot 1 = 1$

$\Rightarrow \mathbb{E}[ZX] = 1 - 1 = 0$

$\Rightarrow \boxed{\mathbb{E}(ZX) = 0}$

b| Since X is a random variable, by definition we have,

$\{\omega \in \Omega \mid X(\omega) \leq x\} \in \mathcal{F} \ \forall x \in \mathbb{R}$

Now for $Y = -X$,

we need to show $\{\omega \in \Omega \mid Y(\omega) \leq y\} \in \mathcal{F} \ \forall y \in \mathbb{R}$

$\{\omega \in \Omega \mid Y(\omega) \leq y\} = \{\omega \in \Omega \mid -X(\omega) \leq y\}$

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$= \{\omega \in \Omega \mid X(\omega) \geq -y\}$

Note that since $\{\omega \in \Omega \mid X(\omega) \leq x\} \in F \; \forall x \in \mathbb{R}$

Then, $\lim_{\varepsilon < 0, \varepsilon \to 0} \{\omega \in \Omega \mid X(\omega) \leq x + \varepsilon\} \in F$

$\Rightarrow \{\omega \in \Omega \mid X(\omega) < x\} \in F \; \forall x \in F$

Then, consider the set $\{\omega \in \Omega \mid X(\omega) < -y\}$

Since $y \in \mathbb{R}$, $-y \in \mathbb{R}$,

$\Rightarrow \{\omega \in \Omega \mid X(\omega) < -y\} \in F$

By the properties of σ-Algebra, $A \in F \Rightarrow A^c \in F$

$\therefore \{\omega \in \Omega \mid X(\omega) < -y\}^c \in F$

$\Rightarrow \{\omega \in \Omega \mid X(\omega) \geq -y\} \in F$

$\Rightarrow \{\omega \in \Omega \mid Y(\omega) \leq y\} \in F \;\; \forall y \in \mathbb{R}$.

$\therefore$ Y is a Random Variable

c) Now, the distribution function of Y,

$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(-X \leq y)$ since $Y = -X$

$= \mathbb{P}(X \geq -y)$

$= 1 - \mathbb{P}(X < -y)$ since $\{X \geq -y\} = \{X < -y\}^c$

By the Continuity of Measure,

$\lim_{\varepsilon < 0, \varepsilon \to 0} \mathbb{P}(X \leq x + \varepsilon) = \mathbb{P}(X < x)$

$= \{\omega \in \Omega \mid X(\omega) \geq -y\}$

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$$\lim_{\varepsilon < 0, \varepsilon \to 0} \mathbb{P}(X \leq x + \varepsilon) = \mathbb{P}(X < x)$$

$$\Rightarrow F_Y(y) = 1 - \mathbb{P}(X < -y)$$
$$= 1 - \lim_{\varepsilon < 0, \varepsilon \to 0} \mathbb{P}(X \leq -y + \varepsilon)$$
$$= 1 - \lim_{\varepsilon < 0, \varepsilon \to 0} F_X(-y+\varepsilon) \text{ by def. of } F_X$$

$$\Rightarrow \boxed{F_Y(y) = 1 - \lim_{\varepsilon < 0, \varepsilon \to 0} F_X(-y+\varepsilon)} \quad \forall y \in \mathbb{R}$$

d) 2 dice are rolled at the same time in a single trial. To get a sum of 3, either the first die shows 1 and second shows 2 or vice-versa.
i.e., there are two favorable outcomes: $\{(1,2),(2,1)\}$

Since each die has 6 faces, the total number of outcomes of throwing 2 dice is clearly $6 \cdot 6 = 36$.

$\therefore \mathbb{P}(\text{sum of dice} = 3) = \dfrac{2}{36} = \dfrac{1}{18}$ (equally likely outcomes)

and, $\mathbb{P}(\text{not getting sum} = 3) = 1 - \mathbb{P}(\text{getting sum } 3)$
$= 1 - \dfrac{1}{18} = \dfrac{17}{18}$

If twenty independent trials are conducted,

the probability of getting exactly 5 trials with sum 3 is

$$\Rightarrow F_Y(y) = 1 - \mathbb{P}(X < -y)$$
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If twenty independent trials are conducted,

the probability of getting exactly 5 trials with sum 3 is

(binomially distributed)

$\underbrace{\binom{20}{5}}\underbrace{\left(\frac{1}{18}\right)^5}\left(\frac{17}{18}\right)^{15} \rightarrow$ in the remaining 15 trials, we need sum $\neq 3$ with prob. 17/18

# of ways to choose 5 trials out of 20

in 5 trials, we need the sum $= 3$ with prob. $= 1/18$

$\therefore$ desired probability $= \binom{20}{5}\left(\frac{1}{18}\right)^5\left(\frac{17}{18}\right)^{15}$

e) Since $X, Y \sim N(\mu, \sigma^2)$

We want to calculate $\mathbb{P}(X > Y) = \mathbb{P}(Y - X) < 0$

<u>Claim</u>: $Y - X \sim N(0, 2\sigma^2)$

<u>Proof</u>: Let $Z := -X$

Then $Z = g(X) = -X$ where $g(x) = -x$ is a decreasing function.

$$\Rightarrow f_Z(z) = -f_X(g^{-1}(z)) \cdot (g'(z))'$$

$$= f_X(-z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(-z-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-(-\mu))^2}{2\sigma^2}}$$

$$\Rightarrow Z \sim N(-\mu, \sigma^2)$$

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$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-(-\mu))^2}{2\sigma^2}}$$

$$\Rightarrow Z \sim N(-\mu, \sigma^2)$$

Then, by the Convolution Formula (since X,Y independent)

$$f_{Y+Z}(z) = \int_{\mathbb{R}} f_Y(x) \cdot f_Z(z-x)\, dx$$

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(z-x+\mu)^2} dx$$

$$= \int_{\mathbb{R}} \frac{1}{2\pi\sigma^2} \exp\left(-\frac{1}{2\sigma^2}\left(x^2+\mu^2-2x\mu+(z-x)^2 + \mu^2 + 2(z-x)\mu\right)\right) dx$$

$$= \frac{1}{2\pi\sigma^2}\int_{\mathbb{R}} \exp\left(-\frac{1}{\sigma^2}\left(x^2+\mu^2-2x\mu+\frac{z^2}{2}+z\mu-zx\right)\right) dx$$

$$= \frac{1}{2\sqrt{\pi\sigma^2}}\int_{\mathbb{R}} \frac{1}{\sqrt{\pi\sigma^2}} \exp\left(-\frac{1}{\sigma^2}\left(x^2+\mu^2-2x\mu+\frac{z^2}{4}+\frac{z^2}{4} - zx + z\mu\right)\right) dx$$

$$= \frac{\exp\left(-\frac{z^2}{4\sigma^2}\right)}{2\sqrt{\pi\sigma^2}} \int_{\mathbb{R}} \frac{1}{\sqrt{\pi\sigma^2}} \exp\left(-\frac{1}{\sigma^2}\left(x^2+\mu^2-2x\mu + \frac{z^2}{4} - z(x-\mu)\right)\right) dx$$

$$= \frac{\exp(-z^2/4\sigma^2)}{2\sqrt{\pi\sigma^2}} \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{\pi\sigma^2}} \exp\left(\left(x-\mu-\frac{z}{2}\right)^2\right) dx}_{=\ \text{density function of} \sim N\left(\mu+\frac{z}{2}, \frac{\sigma^2}{2}\right)}$$

$$= \frac{e^{-\frac{z^2}{2(2\sigma^2)}}}{\sqrt{2\pi(2\sigma^2)}} \cdot 1 = \text{density function of} \sim N(0, 2\sigma^2)$$

$\therefore$ $Y+Z$ has normal dist. with $(0, 2\sigma^2)$

$\Rightarrow Y-X \sim N(0, 2\sigma^2)$. $\therefore$ Proved.

$$\Rightarrow f_{Y-X}(z) = \frac{1}{\sqrt{4\pi\sigma^2}} e^{-\frac{(z-0)^2}{4\sigma^2}}$$

$$= \frac{1}{\sqrt{4\pi\sigma^2}} e^{-\frac{z^2}{4\sigma^2}}$$

and $\mathbb{P}(Y-X<0) = \int_{-\infty}^{0} f_{Y-X}(z)\,dz$

since $Y-X$ is cts.

But observe that

$$f_{Y-X}(-z) = \frac{1}{\sqrt{4\pi\sigma^2}} e^{-\frac{(-z)^2}{4\sigma^2}} = \frac{1}{\sqrt{4\pi\sigma^2}} e^{z^2/4\sigma^2}$$

$$= f_{Y-X}(z)$$

which implies f_{Y-X} is even.

Since even, we know that $\int_{-\infty}^{\infty} f_{Y-X}(z)\,dz$

$$= 2\int_{-\infty}^{0} f_{Y-X}(z)\,dz$$

by property of density function

$$\Rightarrow 1 = \int_{-\infty}^{\infty} f_{Y-X}(z)\,dz = 2\int_{-\infty}^{0} f_{Y-X}(z)\,dz$$

$$\Rightarrow \frac{1}{2} = \int_{-\infty}^{0} f_{Y-X}(z)\,dz = \mathbb{P}(Y-X<0)$$

$$\Rightarrow \boxed{\frac{1}{2} = \mathbb{P}(X>Y)}$$

QUESTION 2

a) Now, $\mathbb{P}(X_1 \le x) = \mathbb{P}\left(\max(5, \min(Y, 7)) \le x\right)$

Suppose $x < 5$,

$$\{\max(5, \min(Y,7)) \le x\}$$
$$= \{5 \le x \text{ and } \min(Y,7) \le x\}$$

But since $x < 5$,

$$= \emptyset$$

$$\Rightarrow \mathbb{P}(X \le x) = 0 \text{ if } x < 5.$$

Now suppose $5 \le x < 7$

Then, $\{\max(5, \min(Y,7)) \le x\}$

$$=$$

$$\{\min(Y,7) \le x\} \quad \text{since } x \ge 5$$

The complement of this set,

$$\{\min(Y,7) > x\}$$
$$=$$
$$\{Y > x \text{ and } 7 > x\}$$

Since $x < 7$

$$\Rightarrow \{Y > x \text{ and } 7 > x\}$$
$$= \{Y > x\}$$

$$\Rightarrow \mathbb{P}(X_1 \le x) = \mathbb{P}(\min(Y,7) \le x)$$
$$= 1 - \mathbb{P}(\min(Y,7) > x) \quad \text{since complement}$$
$$= 1 - \mathbb{P}(Y > x)$$
$$= \mathbb{P}(Y \le x)$$

Since Y is discretely uniform on $[0, 10]$,

we have $\mathbb{P}(Y \leq x) = \begin{cases} \dfrac{k}{11} & \text{if } k-1 \leq x < k \\ & \text{for } k = 1, 2, \ldots, 9 \\ 1 & \text{if } x \geq 10 \\ 0 & \text{if } x < 0 \end{cases}$

(shown in class)

Since $5 \leq x < 7$,

$$\mathbb{P}(Y \leq x) = \begin{cases} 6/11 & \text{if } 5 \leq x < 6 \\ 7/11 & \text{if } 6 \leq x < 7 \end{cases}$$

Using the distribution above.

Now, suppose $7 \leq x < 10$,

Then $\{\max(5, \min(Y, 7)) \leq x\}$

$= \{\min(Y, 7) \leq x\}$ since $x \geq 7 > 5$

$= \{Y \leq x \text{ or } 7 \leq x\}$ But since $x \geq 7$ by assumption,

this event always occurs.

$\Rightarrow \mathbb{P}(\max\{5, \min(Y, 7)\} \leq x) = 1$ for $x \geq 7$.

$$\therefore \boxed{F_{X_1}(x) = \begin{cases} 0 & \text{if } x < 5 \\ 6/11 & \text{if } 5 \leq x < 6 \\ 7/11 & \text{if } 6 \leq x < 7 \\ 1 & \text{if } x \geq 7 \end{cases}}$$

For X_2, we follow the same steps above but use Z instead of Y when necessary.

Suppose $x < 5$,

then just as above,

$$\mathbb{P}(X_2 \leq x) = \mathbb{P}(\emptyset) = 0$$

Suppose $5 \leq x < 7$,

then just with the same reasoning as above,

$$\mathbb{P}(X_2 \leq x) = \mathbb{P}(Z \leq x)$$

Since, Z is uniformly distributed on $[0, 10]$,

we have $\mathbb{P}(Z \leq x) = \begin{cases} \frac{x}{10} & \text{if } 0 \leq x < 10 \\ 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 10 \end{cases}$

Since, $0 \leq 5 \leq x < 7 < 10$,

$$\mathbb{P}(X_2 \leq x) = \mathbb{P}(Z \leq x) = \frac{x}{10}$$

Suppose $x \geq 7$,

then just like above,

$$\mathbb{P}(X_2 \leq x) = 1$$

$$\therefore \boxed{F_{X_2}(x) = \begin{cases} 0 & \text{if } x < 5 \\ \frac{x}{10} & \text{if } 5 \leq x < 7 \\ 1 & \text{if } x \geq 7 \end{cases}}$$

b) Observe X_1: it is sufficient to show that $\text{Im}\, X_1$ is countable and $\{X_1 = x\} \in \mathcal{F}$ $\forall x \in \mathbb{R}$

For $x < 5$, $\{X_1 \leq x\} = \emptyset$ as shown above which also implies that for any $x < 5$,

$\underline{\{X_1 = x\} = \emptyset \in \mathcal{F}}$

and,

$\therefore$ $\underline{\mathbb{P}(X_1 = x) = 0}$ for $x < 5$ finite.

For $5 \leq x \leq 7$,

$$\{X_1 = \max(5, \min(Y, 7)) = x\}$$
$$\equiv$$
$$\{\min(Y, 7) = x\} \equiv \{Y = x\} \text{ since } x \leq 7$$
$$\Rightarrow \mathbb{P}(X_1 = x) = \mathbb{P}(Y = x)$$

Since Y is discrete, X_1 takes on countably many values in $5 \leq x \leq 7$.

And $\{X_1 = x\} = \{Y = x\} \in \mathcal{F}$ since Y discrete.

If $x > 7$,

Then again, observe that

$$\mathbb{P}(X_1 = x) = \mathbb{P}(\{\max(5, \min(Y, 7)) = x\})$$
$$= \mathbb{P}(\{Y = x \text{ or } 7 = x\})$$
$$= \mathbb{P}(\{Y = x\}) \quad \text{since } x > 7$$
$$= \mathbb{P}(Y = x)$$

$\therefore$ Since Y is discrete, X takes on countably many values in $x > 7$.

and $\{X_1 = x\} = \{Y = x\} \in \mathcal{F}$.

$\Rightarrow X$ takes on countably many values in $\mathbb{R}$

and $\{X_1 = x\} \in F \;\; \forall x \in \mathbb{R}$

$\therefore$ X is discrete.

We claim that X_2 is neither discrete nor continuous.

It is not discrete since for, $5 \leq x < 7$,

$X_2 = \max(5, \min(z, 7)) = z$ which is continuous.

$\therefore$ in $5 \leq x < 7$, X_2 takes on uncountably many values.

It is not continuous since

$$\mathbb{P}(X_2 = 5) = \mathbb{P}(X_2 \leq 5) - \mathbb{P}(X_2 < 5)$$
$$= \frac{5}{10} - 0$$
$$= \frac{1}{2}$$

$\Rightarrow \mathbb{P}(X_2 = 5) \neq 0$.

$\therefore$ X_2 is neither continuous nor discrete.

QUESTION 3

In general, if (X,Y) is uniform in region A,

Then, we know $f_{X,Y}(u,y) = \text{constant}$ in A.

Now, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(u,y)\,du\,dy = 1$ by properties of $f_{X,Y}$.

Since $\underset{\text{uniform}}{(X,Y) \in A}$, $f_{X,Y}(u,y) = 0$ if $(X,Y) \notin A$.

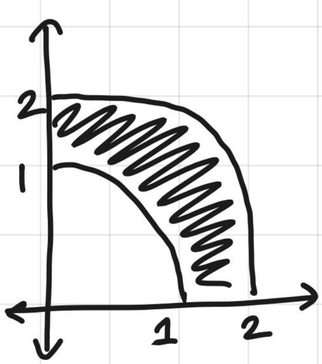
$$\Rightarrow \iint_A f_{X,Y}(u,y)\,du\,dy = 1$$

$$= f_{X,Y}(u,y) \iint_A 1\,du\,dy = 1 \quad \text{since } f_{X,Y} \text{ const. in } A$$

and $\iint_A 1\,du\,dy = \text{area}(A)$

Now, Area(A) can be seen geometrically as

$$\frac{1}{4}\left(\text{Area of circle of radius 2}\right) - \frac{1}{4}\left(\text{Area of circle of radius 1}\right)$$



$$= \frac{1}{4}\left(\pi(2)^2 - \pi(1)^2\right)$$

$$= \frac{1}{4}(4\pi - \pi) = \frac{3\pi}{4}$$

$$\Rightarrow f_{X,Y}(x,y) \cdot \frac{3\pi}{4} = 1$$

$$\Rightarrow \boxed{f_{X,Y}(x,y) = \frac{4}{3\pi}} \quad \forall (x,y) \in A$$

Then, marginal density of X,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,v)\,dv \quad (\text{textbook})$$

Now, we have (in A) $y \geq 0 \Rightarrow y^2 \geq 0$

$$1 \leq x^2 + y^2 \leq 4$$

$$\Rightarrow 1 - x^2 \leq y^2 \leq 4 - x^2 \Rightarrow \max(0, 1-x^2) \leq y^2$$

Then if $x \geq 0$, and $1 - x^2 \geq 0$

$$\Rightarrow 1 \geq x^2 \Rightarrow x \leq 1$$

then, $1 - x^2 \leq y^2 \leq 4 - x^2$

$$\Rightarrow \text{if } 0 \leq x \leq 1,\ f_X(x) = \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \frac{4}{3\pi}\,dy$$

$$\Rightarrow f_X(x) = \frac{4}{3\pi}\left(\sqrt{4-x^2} - \sqrt{1-x^2}\right)$$

Now, suppose $1 - x^2 < 0$, $\Rightarrow x > 1$

then $0 \leq y^2 \leq 4 - x^2$

$$\Rightarrow 0 \leq y \leq \sqrt{4-x^2}$$

and $1 < x^2 \leq 4$

$$\Rightarrow 1 < x \leq 2,$$

$$\therefore \text{if } 1 < x \leq 2,\ f_X(x) = \int_0^{\sqrt{4-x^2}} \frac{4}{3\pi}\,dy$$

$$\Rightarrow f_X(x) = \frac{4}{3\pi}\sqrt{4-x^2} \quad \text{if } 1 < x \le 2$$

$$\Rightarrow f_X(x) = \begin{cases} \frac{4}{3\pi}\left(\sqrt{4-x^2} - \sqrt{1-x^2}\right) & \text{if } 0 \le x \le 1 \\ \frac{4}{3\pi}\left(\sqrt{4-x^2}\right) & \text{if } 1 < x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Now, $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$$= \int_0^1 \frac{4x}{3\pi}\left(\sqrt{4-x^2} - \sqrt{1-x^2}\right)dx$$

$$+ \int_1^2 \frac{4x}{3\pi}\left(\sqrt{4-x^2}\right)dx \quad \text{since } f_X(x) = 0 \text{ otherwise}$$

$$= \int_0^2 \frac{4x}{3\pi}\left(\sqrt{4-x^2}\right)dx - \int_0^1 \frac{4x}{3\pi}\sqrt{1-x^2}\,dx$$

let $u = 4-x^2$ $\quad$ let $t = 1-x^2$

$\Rightarrow du = -2x\,dx$ $\quad$ $\Rightarrow dt = -2x\,dx$

Changing the bounds appropriately,

$$\int_4^0 \frac{-2}{3\pi}\sqrt{u}\,du - \int_1^0 \frac{-2}{3\pi}\sqrt{t}\,dt$$

$$= \frac{2\cdot 2}{3\cdot 3}\left[\left(u\right)^{3/2}\right]_0^4 - \frac{2}{3}\cdot\frac{2}{3}\left[\left(t^{3/2}\right)\right]_0^1$$

$$= \frac{4}{9}\left(\frac{8}{\pi} - \frac{1}{\pi}\right)$$

$$= \frac{28}{9\pi}$$

$$\therefore \boxed{\mathbb{E}X = \frac{28}{9\pi}}$$

QUESTION 4

a) Now $\mathbb{E}[X+Y] = \mathbb{E}(X) + \mathbb{E}(Y)$ by Linearity of Expectation.

Now, $\mathbb{P}(X=x) = \mathbb{P}($first $x-1$ draws were not yellow and x^{th} draw was yellow$)$

Since, there are $n-1$ non-yellow balls,
the probability of picking a non-yellow ball $x-1$ times is $\left(\frac{n-1}{n}\right)^{x-1}$. In the x^{th} draw

we must pick a yellow ball (for the first time)

with Probability $\frac{1}{n}$

$$\text{So, } \mathbb{P}(X=x) = \left(\frac{n-1}{n}\right)^{x-1}\left(\frac{1}{n}\right) \quad \forall x \geq 1$$

$$\text{Then, } \mathbb{E}[X] = \sum_{x=1}^{\infty} x \cdot \mathbb{P}(X=x)$$

$$= \sum_{x=1}^{\infty} x \left(\frac{n-1}{n}\right)^{x-1}\left(\frac{1}{n}\right)$$

$$= \frac{1}{n} \sum_{x=1}^{\infty} x \left(\frac{n-1}{n}\right)^{x-1}$$

We know if $|r| < 1$, then $\sum_{k=0}^{\infty} r^{x} = \frac{1}{1-r}$ by inf. sum of Geometric Series formula

$$\Rightarrow \frac{d}{dr} \sum_{k=0}^{\infty} r^{x} = \frac{d}{dr}\left(\frac{1}{1-r}\right)$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{d}{dr}(r^{x}) = \frac{1}{(1-r)^2}$$ (since derivative is linear)

$$\Rightarrow \sum_{k=0}^{\infty} x r^{x-1} = \frac{1}{(1-r)^2}$$

$$\Rightarrow \sum_{k=1}^{\infty} x r^{x-1} = \frac{1}{(1-r)^2}$$ sine for $x=0$, $x r^{x-1} = 0$

Since $n \geq 1$, $\frac{n-1}{n} < 1$ clearly,

$$\Rightarrow \sum_{k=1}^{\infty} x\left(\frac{n-1}{n}\right)^{x-1} = \frac{1}{\left(1-\frac{n-1}{n}\right)^2}$$

$$= \frac{n^2}{(n-n+1)^2}$$

$$= n^2$$

$$\Rightarrow \frac{1}{n} \sum_{k=1}^{\infty} x\left(\frac{n-1}{n}\right)^{x-1} = \frac{n^2}{n} = n.$$

$\therefore \mathbb{E}[X] = n.$

Similarly, we can show $\mathbb{E}[Y] = n$ by the same logic for red balls.

This implies $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 2n$

$$\therefore \boxed{\mathbb{E}[X+Y] = 2n}$$

X and Y are <u>NOT independent</u> since

$\mathbb{P}(Y=k \mid X=k) = 0 \neq \mathbb{P}(Y=k)$ since if we're given that a yellow ball was drawn for the first time on the k^{th} draw, it is impossible that a red ball can also be drawn on the same draw. $\therefore$ Probability $= 0$.
This is true for any $k \geq 1$.

b| We need to calculate $\mathbb{P}(X=x, Y=y)$
i.e., yellow ball drawn for the first time on x^{th} draw, red ball drawn for the first time on y^{th} draw.

if $x=y$, $\mathbb{P}(X=x, Y=y) = 0$ since different colored balls cannot be drawn on the same draw in a trial.

Suppose $x<y$. We need that in the first $(x-1)$ draws, neither a yellow nor a red ball is drawn; in the x^{th} draw, we draw a yellow ball, in the $(x+1)^{th}$ to $(y-1)^{th}$ draw $\equiv$ the next

This implies $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 2n$

$$\therefore \boxed{\mathbb{E}[X+Y] = 2n}$$

X and Y are <u>NOT independent</u> since

$\mathbb{P}(Y=k \mid X=k) = 0 \neq \mathbb{P}(Y=k)$ since if we're given that a yellow ball was drawn for the first time on the k^{th} draw, it is impossible that a red ball can also be drawn on the same draw. $\therefore$ Probability $=0$.
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b) We need to calculate $\mathbb{P}(X=x, Y=y)$
i.e., yellow ball drawn for the first time on x^{th} draw, red ball drawn for the first time on y^{th} draw.

if $x=y$, $\mathbb{P}(X=x, Y=y) = 0$ since different colored balls cannot be drawn on the same draw in a trial.

Suppose $x<y$. We need that in the first $(x-1)$ draws, neither a yellow nor a red ball is drawn; in the x^{th} draw, we draw a yellow ball, in the $(x+1)^{th}$ to $(y-1)^{th}$ draw $\equiv$ the next

$y-x-1$ draws, we don't draw a red ball and finally in the y^{th} draw, we draw a red ball.

which implies, $\mathbb{P}(X=x, Y=y) = \left(\frac{n-2}{n}\right)^{x-1}\left(\frac{1}{n}\right) \cdot \left(\frac{n-1}{n}\right)^{y-x-1}\left(\frac{1}{n}\right)$

$$= \frac{(n-2)^{x-1}(n-1)^{y-x-1}}{n^y}$$

By an identical logic,
if $y < x$,

$$\mathbb{P}(X=x, Y=y) = \left(\frac{n-2}{n}\right)^{y-1}\left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right)^{x-y-1}\left(\frac{1}{n}\right)$$

$$= \frac{(n-2)^{y-1}(n-1)^{x-y-1}}{n^x}$$

$$\therefore \; p_{X,Y}(x,y) = \begin{cases} 0 & \text{if } x=y \text{ or or } x \le 0 \text{ or } y \le 0 \\ \dfrac{(n-2)^{x-1}(n-1)^{y-x-1}}{n^y} & \text{if } 0 < x < y \\ \dfrac{(n-2)^{y-1}(n-1)^{x-y-1}}{n^x} & \text{if } 0 < y < x \end{cases}$$

QUESTION 5

Let

A_i be the event that the ith Statistics student is paired with a Mathematics student.

for $i = 1, 2, \dots, 20$.

Let $\mathbb{1}_{A_i}$ be the indicator function of A_i

Note that $\mathbb{P}(A_i) = \dfrac{\text{\# of math students}}{\text{\# of total remaining students}}$

$= \dfrac{20}{\underbrace{40-1}} = \dfrac{20}{39}$

↳ since the ith Stats student cannot be paired with him/herself.

Clearly, the number of 1 stats student - 1 math student pairs $= A_1 + A_2 + \dots + A_{20}$

Since $\sum_{i=1}^{20} A_i = m \Leftrightarrow$ some m-subset of the set $\{A_i\}$ occurred

$\Leftrightarrow$ there are m Stats students paired with Math students.

$\Rightarrow$ Expected number of desired pairs

$=$

$\mathbb{E}[\sum_{i=1}^{20} A_i]$

$= \sum_{i=1}^{20} \mathbb{E}[A_i]$ by Linearity of Expectations.

$\Rightarrow$ Expected number of pairs $= 20 \cdot \mathbb{E} A_i$

$$= 20 \cdot \frac{20}{39} = \frac{400}{39}$$

$\therefore$ The expected number of pairs with 1 Stats student and 1 Math student is $\boxed{\frac{400}{39}}$

QUESTION 6

Let $A := \{\text{citizen is wizard}\}$
$B := \{\text{citizen is muggle}\}$
$C_i := \{\text{owlet } i \text{ claims wizard}\}$; $E_i = \{\text{owlet } i \text{ claims muggle}\}$
$D := \{\text{owlets disagree}\}$

a) Now, $\mathbb{P}(D) = \mathbb{P}(D|A) \cdot \mathbb{P}(A) + \mathbb{P}(D|B) \cdot \mathbb{P}(B)$

$\downarrow$

by the Partition Theorem

where we partition on $\{A, B\}$ since a citizen can either be a muggle or a wizard.

We're given that a single owlet says citizen is a wizard if the citizen is indeed a wizard with prob.

$$\mathbb{P}(C_i|A) = \frac{90}{100} \Rightarrow \mathbb{P}(E_i|A) = \frac{10}{100}$$

and $$\mathbb{P}(C_i|B) = \frac{5}{100} \Rightarrow \mathbb{P}(E_i|B) = \frac{95}{100}$$

For the owlets to disagree, they must claim different results.

i.e., $$\mathbb{P}(D|A) = \mathbb{P}(C_1|A)\,\mathbb{P}(E_2|A) + \mathbb{P}(C_2|A) \cdot \mathbb{P}(E_1|A)$$

$$= \frac{9\cancel{0}}{10\cancel{0}} \cdot \frac{1\cancel{0}}{10\cancel{0}} + \frac{1\cancel{0}}{10\cancel{0}} \cdot \frac{9\cancel{0}}{10\cancel{0}}$$

$$= \frac{2 \cdot 9}{100} = \frac{18}{100}$$

Similarly, $\mathbb{P}(D|B) = \mathbb{P}(C_1|B) \cdot \mathbb{P}(E_2|B) + \mathbb{P}(C_2|B) \cdot \mathbb{P}(E_1|B)$

$$= \frac{\cancel{8}}{\cancel{100}_{20}} \cdot \frac{\cancel{95}^{19}}{\cancel{100}_{20}} + \frac{\cancel{95}^{19}}{\cancel{100}_{20}} \cdot \frac{\cancel{8}}{\cancel{100}_{20}}$$

$$= \frac{19}{200}$$

and we're given that $\mathbb{P}(A) = \frac{2}{100}$

$$\Rightarrow \mathbb{P}(B) = \frac{98}{100}$$

$$\therefore \mathbb{P}(D) = \frac{18}{100} \cdot \frac{2}{100} + \frac{19}{\cancel{200}_{100}} \cdot \frac{\cancel{98}^{49}}{100}$$

$$= \frac{967}{10000} = 0.0967$$

$\therefore$ Probability that both owlets disagree $= 0.0967$.

b) We need to find, $\mathbb{P}(A | C_1 \cap C_2)$

i.e., Probability that citizen is wizard given both owlets claim wizard.

By Baye's Theorem, $\mathbb{P}(A | C_1 \cap C_2) = \dfrac{\mathbb{P}(C_1 \cap C_2 | A) \cdot \mathbb{P}(A)}{\mathbb{P}(C_1 \cap C_2 | A) \cdot \mathbb{P}(A) + \mathbb{P}(C_1 \cap C_2 | B) \cdot \mathbb{P}(B)}$

Similarly, $\mathbb{P}(D|B) = \mathbb{P}(C_1|B) \cdot \mathbb{P}(E_2|B) + \mathbb{P}(C_2|B) \cdot \mathbb{P}(E_1|B)$

$$= \frac{\cancel{8}}{\cancel{100}_{20}} \cdot \frac{\cancel{95}^{19}}{\cancel{100}_{20}} + \frac{\cancel{95}^{19}}{\cancel{100}_{20}} \cdot \frac{\cancel{8}}{\cancel{100}_{20}}$$

$$= \frac{19}{200}$$

and we're given that $\mathbb{P}(A) = \frac{2}{100}$

$$\Rightarrow \mathbb{P}(B) = \frac{98}{100}$$

$$\therefore \mathbb{P}(D) = \frac{18}{100} \cdot \frac{2}{100} + \frac{19}{\cancel{200}_{100}} \cdot \frac{\cancel{98}^{49}}{100}$$

$$= \frac{967}{10000} = 0.0967$$

$\therefore$ Probability that both owlets disagree $= 0.0967$.

b) We need to find, $\mathbb{P}(A \mid C_1 \cap C_2)$

i.e., Probability that citizen is wizard given both owlets claim wizard.

By Baye's Theorem, $\mathbb{P}(A \mid C_1 \cap C_2) = \dfrac{\mathbb{P}(C_1 \cap C_2 | A) \cdot \mathbb{P}(A)}{\mathbb{P}(C_1 \cap C_2 | A) \cdot \mathbb{P}(A) + \mathbb{P}(C_1 \cap C_2 | B) \cdot \mathbb{P}(B)}$

where we partition on $\{A, B\}$

Now $\mathbb{P}(C_1 \cap C_2 | A) =$ probability that both owlets claim he's a wizard given that he is indeed a wizard.

Since owlets detect independently,

$$= \underbrace{\frac{90}{100}}_{} \cdot \underbrace{\frac{90}{100}}_{} \quad \text{(given)}$$

probability of each owlet detecting wizard if actually wizard.

Similarly, $\mathbb{P}(C_1 \cap C_2 | B) = \frac{5}{100} \cdot \frac{5}{100}$ (given)

$$\therefore \mathbb{P}(A | C_1 \cap C_2) = \frac{\frac{90}{100} \cdot \frac{90}{100} \cdot \frac{2}{100}}{\frac{90}{100} \cdot \frac{90}{100} \cdot \frac{2}{100} + \frac{5}{100} \cdot \frac{5}{100} \cdot \frac{98}{100}}$$

$$= \frac{90 \cdot 90 \cdot 2}{90 \cdot 90 \cdot 2 + 5 \cdot 5 \cdot 98}$$

$$= \frac{16200}{18650}$$

$$= \frac{324}{373} \approx 0.868633$$

$\therefore$ Probability of citizen being a wizard given that both owlets claim he's a wizard $= \frac{324}{373}$.

≈ 0.86863

QUESTION 7

a) First, we find $f_X(x)$ the marginal density function of X.

Now, we know $f_X(x) = \int_{-\infty}^{\infty} f(x, v)\, dv$ (textbook)

$$\Rightarrow f_X(x) = \int_0^1 \frac{2}{4+\pi}(v + \sin x)\, dv$$

if $0 < x < \pi$

since $f(x,y) = 0$ otherwise.

$$\Rightarrow f_X(x) = \frac{2}{4+\pi}\left(\frac{v^2}{2} + v\sin x\right)\Bigg|_0^1$$

$$= \frac{2}{4+\pi}\left(\frac{1}{2} + \sin x\right)$$

$$\therefore f_X(x) = \begin{cases} (2/4+\pi)\left(\frac{1}{2} + \sin x\right) & \text{if } 0 < x < \pi \\ 0 & \text{otherwise} \end{cases}$$

Then, we know $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)}$ s.t. $f_X(x) \neq 0$

$$\Rightarrow f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$$

if $0 < x < \pi$

since then $f_X(x) \neq 0$

Now if $0 \geq y$ or $1 \leq y$, $f(x,y) = 0$

$$\Rightarrow f_{Y|X}(y|x) = 0 \quad \text{if} \quad 0 \geq y \text{ or } y \geq 1$$

and if $0 < y < 1$, $f(x,y) = \frac{2}{4+\pi}(y + \sin x)$

$$\Rightarrow f_{Y|X}(y|x) = \frac{\cancel{\frac{2}{4+\pi}}(y+\sin x)}{\cancel{\frac{2}{4+\pi}}\left(\frac{1}{2} + \sin x\right)}$$

$$= \frac{2(y+\sin x)}{1+2\sin x}$$

$$\Rightarrow f_{Y|X}(y|x) = \begin{cases} \frac{2(y+\sin x)}{1+2\sin x} & \text{if } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

for $0 < x < \pi$ where it's defined.

b) We show that they are uncorrelated i.e., $\operatorname{cov}(X,Y) = 0$

Now, $\operatorname{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}X \cdot \mathbb{E}Y$

$$\Rightarrow f_{Y|X}(y|x) = 0 \quad \text{if } 0 \geq y \text{ or } y \geq 1$$

and if $0 < y < 1$, $f(x,y) = \frac{2}{4+\pi}\left(y + \sin x\right)$

$$\Rightarrow f_{Y|X}(y|x) = \frac{\cancel{\frac{2}{4+\pi}}\left(y+\sin x\right)}{\cancel{\frac{2}{4+\pi}}\left(\frac{1}{2}+\sin x\right)}$$

$$= \frac{2(y+\sin x)}{1+2\sin x}$$

$$\Rightarrow f_{Y|X}(y|x) = \begin{cases} \frac{2(y+\sin x)}{1+2\sin x} & \text{if } 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

for $0 < x < \pi$ where it's defined.

b| We show that they are uncorrelated

i.e., $\text{cov}(X,Y) = 0$

Now, $\text{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}X \cdot \mathbb{E}Y$

Now, $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$$= \int_0^{\pi} \frac{2x}{4+\pi}\left(\frac{1}{2} + \sin x\right) dx \quad \text{since } f_X(x) = 0 \text{ otherwise}$$

$$= \frac{2}{4+\pi}\left[\int_0^{\pi} \frac{x}{2}\,dx + \int_0^{\pi} x \sin x\,dx\right]$$

$$= \frac{2}{4+\pi}\left[\left[\frac{x^2}{4}\right]_0^{\pi} + \left[-x\cos x\right]_0^{\pi} + \int_0^{\pi} \cos x\,dx\right]$$

(Integration by Parts)

$$= \frac{2}{4+\pi}\left[\frac{\pi^2}{4} + \pi + \left[\sin x\right]_0^{\pi}\right]$$

$$= \frac{2}{4+\pi}\left[\frac{\pi^2}{4} + \pi\right]$$

Now, $f_Y(y) = \int_{-\infty}^{\infty} f(u,y)\,du$

$$= \int_0^{\pi} \frac{2}{4+\pi}(y + \sin u)\,du \quad \text{if } 0 < y < 1$$

$$= \frac{2}{4+\pi}\int_0^{\pi} y + \sin u\,du$$

$$= \frac{2}{4+\pi}\left[\left[yu\right]_0^{\pi} + \left[-\cos u\right]_0^{\pi}\right]$$

$$= \frac{2}{4+\pi}\left[\pi y + \left[1 - (-1)\right]\right] = \frac{2}{4+\pi}(2 + \pi y)$$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{2}{4+\pi}(2+\pi y) & \text{if } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then, $\mathbb{E}[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy$

$$= \int_0^1 y\left(\frac{2}{4+\pi}(2+\pi y)\right) dy \quad \text{since } f_Y(y) = 0 \text{ otherwise}$$

$$= \frac{2}{4+\pi}\int_0^1 (2y + \pi y^2)\,dy$$

$$= \frac{2}{4+\pi}\left(y^2 + \frac{\pi y^3}{3}\right)\Big|_0^1$$

$$= \frac{2}{4+\pi}\left(1 + \frac{\pi}{3}\right)$$

Finally, $\mathbb{E}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy$ by Law of Subconscious Statistician

$$= \int_0^{\pi}\int_0^1 xy\left(\frac{2}{4+\pi}\right)(y+\sin x)\,dy\,dx \quad \text{since } f(x,y) = 0 \text{ otherwise}$$

$$= \frac{2}{4+\pi}\int_0^{\pi} x \int_0^1 (y^2 + y\sin x)\,dy\,dx$$

$$= \frac{2}{4+\pi}\int_0^{\pi} x\left(\frac{y^3}{3} + \frac{y^2}{2}\sin x\right)\Big|_0^1 dx$$

$$= \frac{2}{4+\pi}\int_0^{\pi} x\left(\frac{1}{3}+\frac{1}{2}\sin x\right)dx$$

$$= \frac{2}{4+\pi}\left[\left(\frac{x^2}{6}\right)_0^{\pi}+\frac{1}{2}\int_0^{\pi} x\sin x\,dx\right]$$

$$= \frac{2}{4+\pi}\left[\frac{\pi^2}{6}+\frac{1}{2}\left[\left(-x\cos x\right)_0^{\pi}+\int_0^{\pi}\cos x\,dx\right]\right]$$

(Integration by parts)

$$= \frac{2}{4+\pi}\left(\frac{\pi^2}{6}+\frac{1}{2}\left(\pi+\left[\sin x\right]_0^{\pi}\right)\right)$$

$$= \frac{2}{4+\pi}\left(\frac{\pi^2}{6}+\frac{\pi}{2}\right)$$

Then, $\operatorname{cov}(X,Y) = \frac{2}{4+\pi}\left(\frac{\pi^2}{6}+\frac{\pi}{2}\right) - \left(\frac{2}{4+\pi}\right)\left(1+\frac{\pi}{3}\right)\left(\frac{2}{4+\pi}\right)\left(\frac{\pi^2}{4}+\pi\right)$

$$= \frac{2}{4+\pi}\left(\frac{\pi^2}{6}+\frac{\pi}{2}-\left(\frac{\pi^2}{4}+\pi\right)\left(\frac{2}{4+\pi}\right)\left(1+\frac{\pi}{3}\right)\right)$$

$$= \frac{2}{4+\pi}\left(\frac{\pi^2}{6}+\frac{\pi}{2}-\left(\frac{\pi^2}{4}+\pi\right)\left(\frac{2}{4+\pi}+\frac{2\pi}{3(4+\pi)}\right)\right)$$

$$= \frac{\pi}{4+\pi}\left(\frac{\pi}{3}+1-\left(\frac{\pi}{4}+1\right)\left(\frac{4}{4+\pi}+\frac{4\pi}{12+3\pi}\right)\right)$$

$$= \frac{\pi}{4+\pi}\left(\frac{\pi}{3}+1-\left(\frac{\pi+4}{4}\right)\left(\frac{4}{4+\pi}+\frac{4\pi}{3(4+\pi)}\right)\right)$$

$$= \frac{\pi}{4+\pi}\left(\frac{\pi}{3}+1-\left(\frac{4}{4}\right)\left(1+\frac{\pi}{3}\right)\right)$$

$$= \frac{\pi}{4+\pi}\left(\frac{\pi}{3}+1-1-\frac{\pi}{3}\right)=0$$

$\therefore$ Since $\operatorname{cov}(X,Y)=0$, X,Y are uncorrelated.

Next, note that since

$$f_{Y|X}(y|x) = \begin{cases} \dfrac{2(y+\sin x)}{1+2\sin x} & \text{if } 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

for $0<x<\pi$ where it's defined.

and, $f_Y(y) = \begin{cases} \dfrac{2}{4+\pi}(2+\pi y) & \text{if } 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

then, clearly $f_{Y|X}(y|x) \neq f_Y(y)$

which implies X,Y are not independent.

(contrapositive)

QUESTION 8

Since $\{A_i\}$ independent, we have

$$\mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) = \lim_{n\to\infty} \prod_{i=1}^{n} \mathbb{P}(A_i)$$

$$= \prod_{i=1}^{\infty} p$$

since $\mathbb{P}(A_i) = p \ \forall j = 1, 2, \ldots$

Now, if $p = 1$,

$$\mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = \prod_{i=1}^{\infty} 1 = 1 \quad \text{(trivially)}$$

if $0 \le p < 1$ (since prob. are non-negative)

$$\Rightarrow \mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \prod_{i=1}^{n} p = \lim_{n\to\infty} p^n$$

Since, $0 \le p < 1$, we know that $\lim_{n\to\infty} p^n = 0$

(from analysis)

$$\Rightarrow \mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = 0$$

$$\therefore \mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = \begin{cases} 1 & \text{if } p = 1 \\ 0 & \text{otherwise} \end{cases}$$