1. (5 points) Answer the following by circling the most appropriate response.

i. What is the sample space of an experiment?

(a) The most likely event
(b) A number between 0 and 1
(c) The set of all possible outcomes ← Ω
(d) A galaxy far far away

ii. Which of the following is an example of a random variable?

(a) The outcome (1 through 6) of a die roll ← # assigned to outcome
~~(b)~~ The outcome (Heads or Tails) of a coin flip
(c) Being dealt a royal flush in poker
(d) Rolling an odd number on a die
(e) All of the above

iii. Let A and B be two events. Which of the following is true about $\mathbb{P}(A|B)$ and $\mathbb{P}(B|A)$?

~~(a)~~ They are always equal
~~(b)~~ They are never equal
~~(c)~~ They are equal when events A and B are independent
(d) They are equal when events A and B are disjoint

$\mathbb{P}(A|B) = 0$ disjoint
$\mathbb{P}(B|A) = 0$

(e) None of the above

iv. You flip a fair coin 10 times, and it lands Heads each time. What is the probabilty it lands Heads on the next flip?

(a) 1
(b) 0.5 ← independent of previous flips
(c) $(0.5)^{11}$
(d) $10(0.5)^{11}$
(e) None of the above

v. Which of the following is *incorrect*?

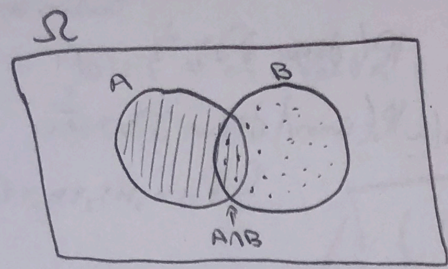
(a) For any two events A and B, $\mathbb{P}(A \cap B) \leq \mathbb{P}(A)$.
(b) For any two random variables X and Y, $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$.
(c) If $\mathbb{P}(A) = 0$ then the event A cannot occur.
(d) For any two events A and B, $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$.
(e) None of the above

consider choosing 0.5 from the interval of real numbers $[0,1]$.

$\mathbb{P}(0.5) = 0$, but it can still happen

2. (15 points)

(a) Explain (briefly!) by drawing a Venn Diagram why $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.



$\mathbb{P}(A \cup B)$ is the probability the event is in either of the two circles.

$\mathbb{P}(A) + \mathbb{P}(B)$ counts the middle part twice, so we subtract it once.

(b) If events A and B are independent, what must $\mathbb{P}(A \cap B)$ be equal to?

If A, B independent, $\mathbb{P}(A|B) = \mathbb{P}(A)$

$\mathbb{P}(A \cap B) = \boxed{\mathbb{P}(A)\mathbb{P}(B)} \leftarrow$

$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \leftarrow$ conditional probability

$\mathbb{P}(A|B)\mathbb{P}(B) = \mathbb{P}(A \cap B)$

independent $\mathbb{P}(A)\mathbb{P}(B) = \mathbb{P}(A \cap B)$

(c) Suppose events A and B are independent. Prove that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)\mathbb{P}(A^c)$.
(Hint: Start with the union law).

*Don't start w/ what you're trying to prove!

$\cancel{\mathbb{P}(A \cup B)} = \mathbb{P}(A) + \mathbb{P}(B)\mathbb{P}(A^c)$

definition of complement

$= \mathbb{P}(A) + \mathbb{P}(B)[\mathbb{P}(\Omega \setminus A)]$

$A \subseteq \Omega$

$= \mathbb{P}(A) + \mathbb{P}(B)[\mathbb{P}(\Omega) - \mathbb{P}(A)]$

$\mathbb{P}(\Omega) = 1$

$= \mathbb{P}(A) + \mathbb{P}(B)[1 - \mathbb{P}(A)]$

Distributive property

$= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B)$

A, B independent - Part (b)

$= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

Part (a)

$\mathbb{P}(A \cup B) = \mathbb{P}(A \cup B)$ ✓

3. (15 points) Two marbles labeled 1 and 2, are in a bag; you draw one at random. You then flip a number of coins equal to the amount that appeared on the marble (i.e. if you drew a 2, you now flip 2 coins, otherwise you flip just one coin).

(a) Suppose you win this game if you do not flip any Heads. What is the probability of winning this game?

Assumes you are equally likely to draw a 1 or 2. → $\mathbb{P}(\text{draw } 1) = \frac{1}{2}$, $\mathbb{P}(\text{draw } 2) = \frac{1}{2}$

Assumes you are equally likely to flip a heads or tails → $\mathbb{P}(\text{win} \mid \text{draw } 1) = \frac{1}{2}$, $\mathbb{P}(\text{win} \mid \text{draw } 2) = \frac{1}{4}$

$\{H, T\}$ $\quad$ $\{HH, HT, TH, TT\}$

$$\mathbb{P}(\text{win}) = \mathbb{P}(\text{win} \cap \text{draw } 1) + \mathbb{P}(\text{win} \cap \text{draw } 2)$$
$$= \mathbb{P}(\text{win} \mid \text{draw } 1)\mathbb{P}(\text{draw } 1) + \mathbb{P}(\text{win} \mid \text{draw } 2)\mathbb{P}(\text{draw } 2)$$
$$= \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{4} + \frac{1}{8} = \frac{2}{8} + \frac{1}{8} = \boxed{\frac{3}{8}}$$

(b) Let p be your answer to (a). Your friend Mr. Bin Omial loves this game, so he invites five people to play. What is the probability exactly three out of the five people win the game? (You may leave this in terms of p if you wish.)

$$p_X(3) = \boxed{\binom{5}{3} p^3 (1-p)^2}$$

(c) If five people play this game, find the expected number of people who will win the game. (You may leave this in terms of p if you wish.)

$$\mathbb{E}[X] = \sum_x x\, p_X(x) \rightarrow \boxed{\mathbb{E}[X] = \sum_{k=1}^{5} k \binom{5}{k} p^k (1-p)^{5-k}}$$

$(= 5p)$

4. (15 points) You have a standard deck of 52 cards (13 ranks (Ace through King), with 4 suits of each). You shuffle the deck thoroughly.

(a) A *pairless hand* is a hand that doesn't contain any pairs of the same rank (i.e. each card is a different rank). What is the probability that you are dealt a pairless hand when you are dealt five cards at random?

total # of hands = $\binom{52}{5}$

total # of hands w/o pairs = $\underbrace{\binom{13}{5}}_{\text{pick 5 values}}\underbrace{\binom{4}{1}^5}_{\text{each card can be of any suit}}$

$$\boxed{\frac{\binom{13}{5}\binom{4}{1}^5}{\binom{52}{5}}}$$ ✓

✓ (b) In poker, the ranks of the cards are ordered from lowest to highest as 2,3,4,..., 10, Jack, Queen, King, Ace. If you are dealt five cards at random, what is the probabilty that the first card you are dealt has strictly higher value than the second card you are dealt?

total # of two-card combinations = 52×51

first card ⇒ $|\{2\}| = |\{3\}| = \dots = |\{Ace\}| = 4$ ← 4 of each card per deck

# of lower values times 4 suits

win → $|\{win|2\}| = 0, |\{win|3\}| = 4, |\{win|4\}| = 8, \dots, |\{win|Ace\}| = 48$

each could be of any suit

$$\boxed{P(win) = \frac{\text{\# of ways to win}}{\text{\# of 2 card hands}} = \frac{4(4+8+\dots+48)}{52\times 51} = \frac{4\sum_{i=1}^{12}(4i)}{52\times 51}}$$

✓ (c) Suppose a magician has a regular deck and aslo a "trick deck" for which drawing an Ace has probability 0.25. Suppose he has a bag that contains 4 regular decks and 1 trick deck, and he randomly draws a deck out the bag. From that deck, he randomly draws a card. Given that card in an Ace, what is the probability the deck was the trick deck?

$T = \{\text{trick deck}\}$, $A = \{Ace\}$, $N = \{\text{normal deck}\}$

$P(A|T) = \frac{1}{4}$, $P(A|N) = \frac{1}{13}$, $P(T) = \frac{1}{5}$, $P(N) = \frac{4}{5}$

$P(T|A) = ?$

Bayes' Rule

Law of total probability

$$P(T|A) = \frac{P(A|T)P(T)}{P(A)} = \frac{P(A|T)P(T)}{P(A\cap T) + P(A\cap N)}$$

convert using conditional probability

$$= \frac{P(A|T)P(T)}{P(A|T)P(T) + P(A|N)P(N)} = \frac{(\frac{1}{4})(\frac{1}{5})}{(\frac{1}{4})(\frac{1}{5}) + (\frac{1}{13})(\frac{4}{5})}$$

$$\boxed{P(T|A) = \frac{(\frac{1}{4})(\frac{1}{5})}{(\frac{1}{4})(\frac{1}{5}) + (\frac{1}{13})(\frac{4}{5})}}$$