1. *(15 points)* Let a and b be positive integers with $a \le b$, and let X be a random variable that takes as values, with equal probability, the powers of 2 in the interval $[2^a, 2^b]$. Find $\mathrm{Var}(2X+1)$.

$$P_X(2^k) = \begin{cases} \frac{1}{b-a+1} & \text{if } k \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \sum_{i=a}^{b} 2^i \frac{1}{b-a+1}$$

$2X+1$ is linear, $\mathbb{E}[(2X+1)] = 2\mathbb{E}[X] + 1$

$$= 1 + \sum_{i=a}^{b} 2^i \frac{2}{b-a+1}$$

$$\mathbb{E}[(2X+1)^2] = \sum_{i=a}^{b} (2\cdot 2^i + 1)^2 \frac{1}{b+a+1}$$ // $P_{2X+1}(2x+1)$ stay the same because one x correspond to one $2x+1$

$$\mathrm{Var}(2X+1) = \mathbb{E}[(2X+1)^2] - (\mathbb{E}[2X+1])^2$$

$$= \sum_{i=a}^{b} (2\cdot 2^i + 1)^2 \frac{1}{b+a+1} - \left(1 + \sum_{i=a}^{b} 2^i \frac{2}{b-a+1}\right)^2$$

parameter

2. *(15 points)* Let X be a Poisson random variable with $\lambda > 0$. Find $\mathbb{E}[\frac{2}{X+1} + 3]$.

$e^{-\lambda} \frac{\lambda^k}{k!}$

$(m-3)(k+1) = 2$

$k+1 = \frac{2}{m-3}$

$k = \frac{2}{m-3}$

let $Y = g(X) = \frac{2}{X+1} + 3$

$P_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}$ $\quad P_Y(m) = \sum_{\{k \mid \frac{2}{k+1}+3=m\}} e^{-\lambda} \frac{\lambda^k}{k!} =$

$$\mathbb{E}\left[\frac{2}{X+1} + 3\right] = \sum_{k=1}^{\infty} \left(\frac{2}{k+1} + 3\right) e^{-\lambda} \frac{\lambda^k}{k!}$$

X

-1

-3

3. *(15 points)* Suppose that n people throw their hats in a box and then each picks one hat at random. (Each hat can be picked by only one person, and each assignment of hats to people is equally likely). What is the expected value of X, the number of people that get back their own hat?

Let H_i be the r.v. that takes 1 if the hat is picked by the right person, and 0 otherwise

$$X = H_1 + H_2 + \dots + H_n$$

$$\mathbb{E}[X] = \mathbb{E}[H_1 + H_2 + \dots + H_n]$$

$$\mathbb{E}[X] = \mathbb{E}[H_1] + \mathbb{E}[H_2] + \dots + \mathbb{E}[H_n] \quad // \text{Linearity}$$

$$= 1 \cdot \frac{1}{n} + 0 \cdot (1 - \frac{1}{n}) + \dots + 1 \cdot \frac{1}{n} + 0 \cdot (1 - \frac{1}{n}) \quad // n \text{ times}$$

$$= \frac{1}{n} \cdot n$$

$$= 1$$

4. *(10 points).* State whether each of the following is true or false. No explanation is required.

(a) If X_1, X_2, ..., X_n are not independent, then it is not true that $\mathbb{E}[X_1 + X_2 + ... + X_n] = \mathbb{E}[X_1] + [X_2] + ... + [X_n]$.

False

(d) If X_1, X_2 ..., X_n are not independent, then it is not true that $\mathrm{Var}(X_1 + X_2 ... + X_n) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + ... + \mathrm{Var}(X_n)$.

True

?

(c) For random variables X and Y, as long as we are given the marginal PMFs, p_X and p_Y, then it is sufficient for us to obtain the joint PMF, $p_{X,Y}$.

False

(d) For random variables X and Y, as long as we are given the joint PMF $p_{X,Y}$, then it is sufficient for us to obtain the marginal PMFs, p_X and p_Y

True

(e) There exists a random variable X with $\mathbb{E}[X] = 5$ and $\mathbb{E}[X^2] = 2$.

True

-2

P. 134 #34

5. *(15 points)* Consider repeatedly and independently tossing a coin with probability of heads being p, $0 < p < 1$. Let X be the number of tosses needed for the the first head to come up. Given that the second head occurred on the n-th toss, what is the conditional PMF of X?

$A = \{$the second head occurred on the n-th toss$\}$

$X \in \{1, \ldots, n-1\}$

$P_X(k) = (1-p)^{k-1} p (1-p)^{n-1-k}$

$= (1-p)^{n-2} p$? ...

-15

6. *(15 points)*. Consider an experiment that has exactly three possible outcomes that occur equally likely. Suppose that n independent repetitions of the experiment are made, and let X_i denote the number of times that outcome i occurs (i=1, 2, 3).

(a) Compute $\mathbb{E}(X_2|X_1 + X_2 = m)$.

$$\mathbb{E}(X_2|X_1+X_2=m) = \sum_{x_2=0}^{m} x_2 \mathbb{P}_{X_2}(X_2=x_2 | X_1+X_2=m)$$

$$= \sum_{x_2=0}^{m} x_2 \binom{m}{x_2}\left(\frac{1}{2}\right)^{x_2}\left(1-\frac{1}{2}\right)^{m-x_2}$$ // equally likely, now we have $X_1+X_2=$

... ?

= ?

-3

(b) Compute $\mathbb{E}(X_2)$.

$$\mathbb{E}(X_2) = \sum_{x_2=0}^{n} x_2 \mathbb{P}_{X_2}(x_2)$$

$$= \sum_{x_2=0}^{n} x_2 \binom{n}{x_2}\left(\frac{1}{3}\right)^{x_2}\left(\frac{2}{3}\right)^{n-x_2}$$ // three equally likely possibility.

?

= ?

-2

For Problems 7 and Problem 8, you can choose one and only one to work out. If you show work on both, we will grade only Problem 7 by default.

7. *(10 points)* Bob and Alice each has a fair coin. Alice tosses her coin till she sees the first head. Bob tosses his coin till he sees the first head. Whoever has made more tosses, we record that number and call it X. Compute the expectation of X. Assume that all the coin tosses are independent.

~~Let A be the r.v. that is the number Alice has tossed~~

~~B be . . . . . . Bob . . .~~

~~$X = \max(A, B)$~~

8. *(10 points)* Toss a fair coin independently 100 times, and let N be the total number of heads. Now toss a second coin, which is biased with the probability $1/N$ of heads and $1 - 1/N$ of tails, independently for N^2 times. Find the expectation of total number of heads that you will obtain when you toss the second (biased) coin.

Let H be the r.v. that is the number of heads obtained from the second coin.

$$\mathbb{E}[H] = \sum_{n=0}^{100} \mathbb{P}_N(n)\,\mathbb{E}[H \mid N=n]$$

$$= \sum_{n=0}^{100} \binom{100}{n}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{100-n} \sum_{h=0}^{n^2} h \binom{n^2}{h}\left(\frac{1}{n}\right)^h\left(1-\frac{1}{n}\right)^{n^2-h}$$

. . . . ?

=

−4