Midterm 1

Math 170A, 2017 Winter.

You have 50 minutes.

All of the policies and guidelines on the class webpages are in effect on this exam.

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **If you use a theorem or proposition from class or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will receive no credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will however still receive partial credit.

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"By signing here, I certify that I have completed this examination while refraining from cheating and adhered to the UCLA Community Standard."

Signature: Zhouheng Sun

1. 15 2. 15 3. 10 4. 14 5. 14 6 or 7. 5 (points)

Total Score 73

1. *(15 points)* Two boxes each contains seven balls. In each box we label the seven balls with the numbers 1, 2, 3, 4, 5, 6 and 7. Draw one ball randomly out of each box.

(a) Find the probability that the two balls are labelled with different numbers.

# of A : $Perm(7,2) = \frac{7!}{5!} = 7 \times 6 = 42$

of Ω : $7 \times 7 = 49$, with uniform probability

$\therefore P(A) = \frac{|A|}{|\Omega|} = \frac{42}{49} = \frac{6}{7}$.

(b) Given that the balls are labelled with different numbers, find the conditional probability that the numbers on the balls result in a sum of 4 or less.

Let $A = \{$sum of the numbers $\le 4\}$, $B = \{$balls are labelled w/ different numbers$\}$.

With Bayes' Rule, $\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A) \cdot \mathbb{P}(A)}{\mathbb{P}(B)}$

Now, $A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1)\}$

$\therefore \mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{6}{49}$.

$B|A = \{(1,2), (1,3), (2,1), (3,1)\}$.

$\therefore \mathbb{P}(B|A) = \frac{|B|A|}{|A|} = \frac{4}{6} = \frac{2}{3}$

$\therefore \mathbb{P}(A|B) = \frac{\frac{2}{3} \cdot \frac{6}{49}}{\frac{6}{7}} = \frac{\frac{4}{49}}{\frac{6}{7}} = \frac{4}{42} = \frac{2}{21}$.

(c) Given that the balls result in a sum of 13 or less, find the conditional probability that the balls were labelled with the same number.

Let $A = \{$balls have the same #$\}$. $B = \{$balls have a sum $\le 13\}$

By Bayes' Rule, $\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\,\mathbb{P}(A)}{\mathbb{P}(B)}$

Now, $\mathbb{P}(A) = 1 - \mathbb{P}(\{$balls have diff #$\}) = 1 - \frac{6}{7} = \frac{1}{7}$ (as shown in (a))

$B^c = \{(7,7)\}$ $\therefore$ $\mathbb{P}(B) = 1 - \mathbb{P}(B^c) = 1 - \frac{|B^c|}{|\Omega|} = 1 - \frac{1}{49} = \frac{48}{49}$.

$A = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7)\}$

$B|A = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$ $\Rightarrow \mathbb{P}(B|A) = \frac{|B|A|}{|A|} = \frac{6}{7}$

$\therefore \mathbb{P}(A|B) = \frac{\frac{6}{7} \cdot \frac{1}{7}}{\frac{48}{49}} = \frac{1}{8}$

2. *(15 points)* You roll a fair four-sided die. If the result is 1 or 2, you roll another fair six-sided die once and stop. If the result is neither, you just stop rolling. Given that the sum total of your rolls is at least 4, what is the conditional probability that the result of the first roll was 1?

$$\mathbb{P}(\text{first}=1 \mid \text{sum} \geq 4) \underset{\text{by Baye's}}{=} \frac{\mathbb{P}(\text{sum} \geq 4 \mid \text{first}=1)\, \mathbb{P}(\text{first}=1)}{\mathbb{P}(\text{sum} \geq 4)} \underset{\text{by total prob. thm}}{=}$$

$$\frac{\mathbb{P}(\text{second roll} \geq 3)\, \mathbb{P}(\text{first}=1)}{\mathbb{P}(\text{sum} \geq 4 \mid \text{first}=1) \cdot \mathbb{P}(\text{first}=1) + \mathbb{P}(\text{sum} \geq 4 \mid \text{first}=2) \cdot \mathbb{P}(\text{first}=2) + \mathbb{P}(\text{sum} \geq 4 \mid \text{first}=3) \cdot \mathbb{P}(\text{first}=3) + \mathbb{P}(\text{sum} \geq 4 \mid \text{first}=4)\, \mathbb{P}(\text{first}=4)}$$

$$= \frac{\frac{4}{6} \cdot \frac{1}{4}}{\frac{4}{6} \cdot \frac{1}{4} + \frac{5}{6} \cdot \frac{1}{4} + 0 + \frac{1}{4}} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{5}{24} + \frac{1}{4}} = \frac{4}{4+5+6} = \frac{4}{15}.$$

3. *(10 points)* Assume that A_1, A_2, A_3, A_4 are independent. Are the events $A_1^c \cap A_3$ and $A_2 \cup A_4$ independent? If it is true, provide a proof. Otherwise give a counter example.

Need to show:

$$\mathbb{P}((A_1^c \cap A_3) \cap (A_2 \cup A_4)) = \mathbb{P}(A_1^c \cap A_3) \cdot \mathbb{P}(A_2 \cup A_4)$$

$$LHS = \mathbb{P}((A_1^c \cap A_3 \cap A_2) \cup (A_1^c \cap A_3 \cap A_4))$$

Axiom of Prob law

$$= \mathbb{P}(A_1^c \cap A_3 \cap A_2) + \mathbb{P}(A_1^c \cap A_3 \cap A_4) - \mathbb{P}(A_1^c \cap A_2 \cap A_3 \cap A_4) \quad (*)$$

By total prob. thm, $\mathbb{P}(A_3) = \mathbb{P}(A_3|A_1)\mathbb{P}(A_1) + \mathbb{P}(A_3|A_1^c)(1-\mathbb{P}(A_1))$

A_1, A_3 are indpdt. $\to$ $= \mathbb{P}(A_1)\mathbb{P}(A_3) + \mathbb{P}(A_3|A_1^c)(1-\mathbb{P}(A_1))$

$\therefore$ Either $\mathbb{P}(A_1) = 1$, or $\mathbb{P}(A_3|A_1^c) = \mathbb{P}(A_3)$.

(1) if $\mathbb{P}(A_1)=1$, then $\mathbb{P}(A_1^c \cap A_3) = \mathbb{P}(A_1^c) + \mathbb{P}(A_3) - \mathbb{P}(A_1^c \cup A_3) \geq 0$

$\therefore \mathbb{P}(A_3) \geq \mathbb{P}(A_1^c \cup A_3)$

However, $A_3 \subset A_1^c \cup A_3$ $\therefore \mathbb{P}(A_3) \leq \mathbb{P}(A_1^c \cup A_3)$

$\therefore \mathbb{P}(A_3) = \mathbb{P}(A_1^c \cup A_3)$

$\therefore \mathbb{P}(A_1^c \cap A_3) = \mathbb{P}(A_3) \cdot \mathbb{P}(A_1^c) = 0$

$\therefore A_1^c$ and A_3 are independent

(2) if $\mathbb{P}(A_3|A_1^c) = \mathbb{P}(A_3)$, then A_1^c and A_3 are independent by definition

$\therefore$ Either way, A_1^c and A_3 are independent, and similarly, for (A_1^c, A_2), (A_1^c, A_4)

So $\mathbb{P}(A_1^c \cap A_3) = (1-\mathbb{P}(A_1))\mathbb{P}(A_3)$.

~~$\mathbb{P}(A_1^c \cap A_3) = \mathbb{P}(A_1^c)\mathbb{P}(A_3) = \mathbb{P}(A_1^c \cap A_3 \cap A_2) + \mathbb{P}(A_1^c \cap A_3 \cap A_2^c)$~~

Let P_n denote $\mathbb{P}(A_n)$.

~~$\mathbb{P}(A_2 \cap A_3) = \mathbb{P}(A_2 \cap A_3 \cap A_1)$~~ $+ \mathbb{P}(A_2 \cap A_3 \cap A_1^c)$

$\therefore \mathbb{P}(A_1^c \cap A_2 \cap A_3) = \mathbb{P}(A_2)P(A_3) - P(A_1)P(A_2)(P(A_3))$

similarly, $\mathbb{P}(A_1^c \cap A_3 \cap A_4) = \mathbb{P}(A_3)P(A_4) - P(A_1)P(A_3)P(A_4)$

$P(A_1^c \cap A_2 \cap A_3 \cap A_4) = \mathbb{P}(A_2)\mathbb{P}(A_3)\mathbb{P}(A_4) - \mathbb{P}(A_1) \cdots \mathbb{P}(A_4)$

$\therefore (*) = P_2P_3 - P_1P_2P_3 + P_3P_4 - P_1P_3P_4 + P_1P_2P_3P_4 - P_2P_3P_4$

$RHS = (1-P_1)P_3(P_2 + P_4 - P_2P_4) = P_2P_3 - P_1P_2P_3 + P_3P_4 - P_1P_3P_4 + P_1P_2P_3P_4 - P_2P_3P_4$

$\therefore LHS = (*) = RHS$ $\square$

4. *(15 points)* An ordinary 52-card deck is randomly divided into 4 groups of 13. If the groups are numbered as 1, 2, 3, and 4, then what is the probability that the division is done and the ace of heart is in group 1, the ace of spade in group 2, the ace of club in group 3, and the ace of diamond in group 4 respectively?

$$\mathbb{P}(A) = \frac{\dfrac{48!}{12!\,12!\,12!\,12!}}{\dfrac{52!}{13!\,13!\,13!\,13!}} = \frac{13^4}{52\times 51\times 50\times 49}$$

More explanations are needed

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(Method I)

5. *(15 points)* Suppose A_1, A_2, ..., A_n are independent events with $P(A_i) = p$ for every i, $i = 1, 2, ..., n$, $0 \le p \le 1$.

(a) Find $\mathbb{P}(A_1 \cap A_2)$.

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2) = p^2$$

(b) Find $\mathbb{P}(A_1 \cup A_2)$.

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2)$$
$$= 2p - p^2$$

(b) Find $\mathbb{P}(\cup_{i=1}^n A_i)$.

$$\mathbb{P}(\cup_{i=1}^n A_i) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} p^k$$

? You should give more intermediate steps.

(−1)

(d) What is the probability that at most k events among A_1, ..., A_n have occurred?

$$\mathbb{P} = \sum_{m=0}^{k} \binom{n}{m} p^m (1-p)^{n-m}$$

For Problem 6 and Problem 7, you can choose one and only one to work out. If you show work on both, we will have to grade only Problem 6 by default.

6. *(15 points)* You roll a fair four-sided die. Whenever the result of the roll is 1 or 2, you roll once more. Whenever the result of the roll is neither, you stop rolling. Given that the sum total of your rolls is at least 4, what is the conditional probability that the result of the first roll was 1?

$$\mathbb{P}(\text{first} = 1 \mid \text{sum} \geq 4)$$

$$= \frac{\mathbb{P}(\text{sum} \geq 4 \mid \text{first} = 1)\,\mathbb{P}(\text{first} = 1)}{\mathbb{P}(\text{sum} \geq 4)}$$

$$= \frac{\frac{2}{4} \cdot \frac{1}{4}}{\mathbb{P}(\text{sum} \geq 4 \mid \text{first} = 1)\,\mathbb{P}(f=1) + \mathbb{P}(s \geq 4 \mid f=2)\,\mathbb{P}(f=2) + \mathbb{P}(s \geq 4 \mid f=3)\,\mathbb{P}(f=3) + \mathbb{P}(s \geq 4 \mid f=4)\,\mathbb{P}(f=4)}$$

$$= \frac{\frac{2}{16}}{\frac{2}{16} + \frac{3}{4} \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4}} = \frac{2}{2+3+4} = \frac{2}{9}$$

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$P_k = \frac{1}{3}P_{k+1} + \frac{2}{3}P_{k-1}$

$P_{k+1} - P_k = 3(P_k - P_{k-1})$. 序列

独立

y_1

$P_{k+1} = 3P_k - 2P_{k-1}$

$\lambda^2 - 3\lambda + 2 = 0$ $\lambda_1 = +1$ $\lambda_2 = +2$

7. *(15 points)* A gambler makes a sequence of independent bets. In each bet he wins one dollar with probability 1/3, and loses one dollar with probability 2/3. Initially, the gambler has 60 dollars, and plays untill he either accumulates 100 dollars or has no money left. What is the probability that the gambler will end up with 100 dollars?

$P_n = c_1 + c_2 \cdot 2^n$, $n \geq 2$

$P_{100} = 1$

The probability that the gambler will gain one dollar is $\frac{1}{3} + \binom{3}{1}\frac{2}{3}\cdot\left(\frac{1}{3}\right)^2 + \binom{5}{3}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^3$

$1 = c_1 + c_2 \cdot 2^{100}$

$0 = c_1 + c_2$

$\Rightarrow c_2 = \frac{1}{2^{100}-1}$

$c_1 = \frac{2^{100}-2}{2^{100}-1}$

$P(\text{rem, win}) = \frac{1}{3}$

$1 - P(\text{rem, lose})$

$P_{100} = 1$

$P_{99} = \frac{1}{3}P_{100} + \frac{2}{3}P_{98}$

$P_{60} = \frac{1}{3}P_{61} + \frac{2}{3}P_{59}$

$P_{59} = \frac{1}{3}P_{60} + \frac{2}{3}P_{58}$

$P_m = \frac{1}{3}P_{m+1} + \frac{2}{3}P_{m-1}$

$P_n = \frac{1}{3}P_{n+1} + \frac{2}{3}P_{n-1}$

$\Sigma P = \sum_{n+1}^{m-1} P + \frac{1}{3}P_m + \frac{1}{3}P_{m+1} + \frac{2}{3}P_n + \frac{2}{3}P_{n-1}$

$P_n + P_m = \frac{1}{3}P_m +$

$\frac{1}{3}P_n + \frac{1}{3}P_m$

$\frac{1}{3}P_1 + \frac{2}{3}P_{99} = \frac{1}{3}$

$P_2 = \frac{1}{3}P_3 + \frac{2}{3}P_1$

$P_1 = \frac{2}{3}P_0 + \frac{1}{3}P_2$

$P_0 = 0$

$P_0 + P_1 + P_{99} + P_{100}$

$P_1 + 2P_{99} = 1$

$= \frac{1}{3}(P_{99} + P_{100})$

$+ \frac{2}{3}(P_0 + P_1)$

$\sum_{i=0}^{100} P = \frac{1}{3}\sum_{i=2}^{100} P + \frac{2}{3}\sum_{i=0}^{98} P + 1$

$P_1 + P_{99} + 1$

$= \frac{1}{3}(P_{99} + 1) + \frac{2}{3}P_1$

$\sum_{i=1}^{99} P = \cancel{\frac{1}{3}\sum_{i=2}^{99}} P_i$

$\frac{1}{3}P_1 = \frac{2}{3}$