Math 170A - Midterm 2

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Discussion session: Tuesday 1pm

Disclaimer: You need to justify your calculations.

Problems	Points	Score
1	6	5.5
2	4	3
3	5	3,5
4	3	2,5
Total	18	14.5

Problem 1. (6 points)

(a) (1 point) Assume that X and Y are such that $\mathrm{var}(X) = 2$, $\mathrm{var}(Y) = 3$ and $\mathrm{cov}(X, Y) = 5$. Compute $\mathrm{var}(2X + 3Y)$.

(b) (1 point) For X and Y two independent random variables such that $E[X] = 1$, $E[Y] = 2$, $\mathrm{var}[X] = 2$ and $\mathrm{var}[Y] = 4$, compute $E[(2X + 3Y)^2]$.

(c) (2 points) Assume that X and Y are two independent Bernoulli random variables with parameters $p_1 = \frac{1}{2}$ and $p_2 = \frac{1}{3}$ respectively, and both of them having as range the set $\{0, 1\}$. Show that $Z = XY$ is also a Bernoulli random variable, compute its parameter p_3 and its expected value.

(d) (2 points) Assume that the PMF of a discrete random variable X is given by:

$$p_X(k) = C \cdot \frac{1}{4^k} \text{ for } k \in \{2, 3, 4, \ldots\},$$

for some $C \in \mathbb{R}$. Determine the value of C.

a) $\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y) + 2\mathrm{cov}(X,Y)$ from class.

so $\mathrm{var}(2X+3Y) = \mathrm{var}(2X) + \mathrm{var}(3Y) + 2\mathrm{cov}(2X,3Y)$

$\mathrm{var}(aX) = a^2X$ so $= 4\mathrm{var}(X) + 3^2(\mathrm{var}(Y)) + 2\mathrm{cov}(2X,3Y)$

$\mathrm{cov}(X,Y)$ is bilinear so $2\mathrm{cov}(2X,3Y) = 2 \cdot 2 \cdot 3 \cdot \mathrm{cov}(X,Y)$

so we have $4 \cdot 2 + 3^2 \cdot 3 + 12 \cdot 5 = 8 + 27 + 60 = \boxed{95}$

b) $E[(2X+3Y)^2] = \mathrm{Var}(2X+3Y) + E[(2X+3Y)]^2$ by variance formula,

since X, Y ind. $\mathrm{Var}(2X+3Y) = \mathrm{Var}(2X) + \mathrm{Var}(3Y)$ (from class).

and $E[2X+3Y] = 2E[X] + 3E[Y]$ by linearity of E

So we have $2^2\mathrm{var}(X) + 3^2\mathrm{var}(Y) + (2E[X] + 3E[Y])^2$

$= 2^2 \cdot 2 + 3^2 \cdot 4 + (2 \cdot 1 + 3 \cdot 2)^2 = 8 + 36 + 64 = \boxed{108}$

c) $Z = XY = \begin{cases} 1 & \text{if } X=Y=1 \\ 0 & \text{o/w} \end{cases}$ so $P_3 = P_1 \cdot P_2 = \frac{1}{2} \cdot \frac{1}{3} = \boxed{\frac{1}{6}}$,

by independence

-.5

$E[Z] = 0 \cdot \frac{5}{6} + 1 \cdot \frac{1}{6} = \boxed{\frac{1}{6}}$

d) $\sum_{k=2}^{\infty} P_X(k) = 1$, so $\sum_{k=2}^{\infty} C \cdot \frac{1}{4^k} = 1$, so $\sum_{k=2}^{\infty} \frac{1}{4^k} = \frac{1}{C}$, $\sum_{k=2}^{\infty} \frac{1}{4^k}$ is geometric series with $a_0 = \frac{1}{16}$, and $r = \frac{1}{4}$, so

$\sum_{k=2}^{\infty} \frac{1}{4^k} = \frac{\frac{1}{16}}{1 - \frac{1}{4}} = \frac{\frac{1}{16}}{\frac{3}{4}} = \frac{4 \cdot 1}{3 \cdot 16} = \frac{1}{12} = \frac{1}{C} \Rightarrow \boxed{C = 12}$

5.5

Problem 2. (4 points)

Let Ω be the set of permutations of the set $\{1,\ldots,n\}$. We say that a permutation $\pi = \{\pi(1),\ldots,\pi(n)\}$ has a fixed point if $\pi(k) = k$ for some $k \in \{1,\ldots,n\}$. Let X be the random variable that gives the number of fixed points of a given permutation.

(a) (2 points) Compute the expected value of X. since $X=\sum X_i$ and E is linear

(b) (2 points) Compute the variance of X.

a) $E[X]=\sum_{i=1}^{n} E[X_i]$ for X_i the corresponding # of fixed points for each i, i.e. $X_i = \begin{cases} 1 & \text{if } \pi(i)=i \\ 0 & \text{o/w} \end{cases}$ ✓

so $E[X_i] = 1 \cdot \frac{1}{n} = \frac{1}{n}$, so $E[X] = \sum_{i=1}^{n} \frac{1}{n} = 1$ ✓

b) $Var(X) = \sum_{k \in \mathbb{R}} k^2 p_X(k) = \sum_{k=0}^{n} k^2 p_X(k) - E[X]^2$

$$\underbrace{\sum_{k \in \mathbb{R}} k^2 p_X(k)}_{E[X]^2}$$

$$= \sum_{k=0}^{n} k^2 \cdot \underbrace{\binom{n}{k}}_{\text{\# of ways to choose k fixed points}} \cdot \underbrace{n!\left(\sum_{i=0}^{n-k} \frac{(-1)^i}{i!}\right)}$$

← # of ways to derange the remaining points

3

3,5

Problem 3. (5 points)

The table below gives you the joint PMF of two discrete random variables X and Y.

	X = 0	X = 2	X = 4
Y = 2	1/4	1/6	1/6
Y = 5	1/6	1/8	1/8

(a) (1 point) Compute the marginal PMFs of X and Y.
(b) (2 points) Compute the conditional PMFs $p_{X|Y}$ and $p_{Y|X}$.
(c) (2 points) Compute the probability of the event that $XY^2 - X > 20$.

a) $P_X(k) = \begin{cases} 1/4 + 1/6 & k=0 \\ 1/6 + 1/8 & k=2 \\ 1/6 + 1/8 & k=4 \\ 0 & o/w \end{cases}$ so $\boxed{P_X(k) = \begin{cases} 5/12 & \text{if } k=0 \\ 7/24 & \text{if } k=2 \text{ or } k=4 \\ 0 & o/w \end{cases}}$?

$P_Y(k) = \begin{cases} 1/4 + 1/6 + 1/6 & \text{if } k=2 \\ 1/6 + 1/8 + 1/8 & \text{if } k=5 \\ 0 & o/w \end{cases}$ so $\boxed{P_Y(k) = \begin{cases} 7/12 & \text{if } k=2 \\ 5/12 & \text{if } k=5 \\ 0 & o/w \end{cases}}$

b) $P_{X|Y}(x|y) = \begin{cases} (1/4)/(7/12) & \text{if } x=0, y=2 \\ (1/6)/(7/12) & \text{if } x=2 \text{ or } 4, y=2 \\ (1/6)/(5/12) & \text{if } x=0, y=5 \\ (1/8)/(5/12) & \text{if } x=2 \text{ or } 4, y=5 \\ 0 & o/w \end{cases} = \boxed{\begin{cases} 3/7 & \text{if } x=0, y=2 \\ 2/7 & \text{if } x=2 \text{ or } 4, y=2 \\ 2/5 & \text{if } x=0, y=5 \\ 3/10 & \text{if } x=2 \text{ or } 4, y=5 \\ 0 & o/w \end{cases}}$

?

$P_{Y|X}(y|x) = \begin{cases} (1/4)/(5/12) & \text{if } x=0, y=2 \\ (1/6)/(7/24) & \text{if } x=2 \text{ or } 4, y=2 \\ (1/6)/(5/12) & \text{if } x=0, y=5 \\ (1/8)/(7/24) & \text{if } x=2 \text{ or } 4, y=5 \\ 0 & o/w \end{cases} = \boxed{\begin{cases} 3/5 & \text{if } x=0, y=2 \\ 4/7 & \text{if } x=2 \text{ or } 4, y=2 \\ 2/5 & \text{if } x=0, y=5 \\ 3/7 & \text{if } x=2 \text{ or } 4, y=5 \\ 0 & o/w \end{cases}}$

c) $P(XY^2 - X > 20)$, for this to be true, clearly $X \neq 0$, if $X=2$, then $y=5$, and if $X=4$, $y=5$ so this is equal to $P(\{X=2 \text{ and } Y=5\} \cup \{X=4 \text{ and } Y=5\})$ so the probability is the sum of those boxes so $1/8 + 1/8 = \boxed{1/4}$

2,5

Problem 4. (3 points)

In a game the first contestant chooses a number among 2 and 4 with equal probability. Then for $k \in \{2,4\}$ the number chosen by the first contestant, the second contestant chooses an integer uniformly at random from the set $\{1,\dots,k\}$. Let X be the number chosen by the first contestant and let Y be the number chosen by the second contestant. Compute $E[Y]$.

$$E[Y] = \sum_{k \in R} k \cdot P_Y(k) \qquad R = \{1,2,3,4\},$$

$$P_Y(4) = P_X(4) \cdot \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

$$P_Y(3) = P_X(4) \cdot \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

? $$P_Y(1) = P_X(2) \cdot \frac{1}{2} + P_X(4) \cdot \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

$$P_Y(2) = P_X(2) \cdot \frac{1}{2} + P_X(4) \cdot \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} = \frac{3}{8}$$

So $$E[Y] = 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{8}$$

$$= \frac{3}{8} + \frac{6}{8} + \frac{3}{8} + \frac{4}{8} = \frac{16}{8} = 2$$

So $\boxed{E[Y] = 2}$