Math 170A - Midterm 2

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Discussion session:

Disclaimer: You need to justify your calculations.

Problems	Points	Score
1	6	4
2	5	2.5
3	6	5.5
4	3	2
Total	20	14

4

Problem 1. *(6 points)*

(a) (*2 point*) Assume that X and Y are such that $\mathbb{E}[X^2] = 3$, $\mathbb{E}[X] = \sqrt{2}$, $\mathbb{E}[Y^2] = 4$, $\mathbb{E}[Y] = 1$, and $cov(X, Y) = 5$. Compute $\mathbb{E}[2XY]$ and $var(5X - 3Y)$.

(b) (*2 points*) Let $X : \Omega \to \{0, 1, \dots\}$ be a discrete random variable with PMF p_X that satisfies the following:

$\frac{\pi^2}{6}.$

$$p_X(k) = \frac{4}{k} p_X(k-1) \text{ for } k \in \{1, 2, \dots\}.$$

Compute the expected value of X.

(c) (*2 points*) Assume that the PMF of a discrete random variable X is given by:

$$p_X(k) = C \cdot \frac{1}{k^2} \text{ for } k \in \{1, 2, \dots\},$$

for some $C \in \mathbb{R}$. Determine the value of C.

(a) $\mathbb{E}[2XY] = 2\mathbb{E}[XY]$

Note that $cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

$5 = \mathbb{E}[XY] - \sqrt{2}\,(1)$

$\Rightarrow \mathbb{E}[XY] = 5 + \sqrt{2}$

$\Rightarrow \mathbb{E}[2XY] = 2(5 + \sqrt{2})$

$= 10 + 2\sqrt{2}$

$var(5X - 3Y) = \mathbb{E}[(5X-3Y)^2] - (\mathbb{E}[5X - 3Y])^2$

$= \mathbb{E}[25X^2 + 9Y^2 - 30XY] - (5\mathbb{E}[X] - 3\mathbb{E}[Y])^2$

$= 25\mathbb{E}[X^2] + 9\mathbb{E}[Y^2] - 30\mathbb{E}[XY] - [5\sqrt{2} - 3(1)]^2$

$= 25(3) + 9(4) - 30(5 + \sqrt{2}) - (5\sqrt{2} - 3)^2$

$= 75 + 36 - 150 - \cancel{30\sqrt{2}} - 50 - 9 + \cancel{30\sqrt{2}}$

$= 111 - 209$

$= -98$

(next page for (b) & (c))

(b) $P_X(k) = \frac{4}{k} P_X(k-1) \quad \forall\, k \in \{1, 2, 3, \dots\}$

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k\, P_X(k)$$

$$= \sum_{k=1}^{\infty} k \frac{4}{k} P_X(k-1)$$

$$= 4 \sum_{k=1}^{\infty} P_X(k-1) = ?$$

(c) $P_X(k) = C \cdot \frac{1}{k^2}$ for $k \in \{1, 2, \dots\}$ $\frac{1}{1-r}$

by property of PMF. $\sum_{k=1}^{\infty} C \cdot \frac{1}{k^2} = 1$

$$\Rightarrow C \sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^2 = 1$$

$$\Rightarrow C = ?$$

2.5

$\mathbb{P}(|X-\mu|\geq\lambda\sigma)\leq\frac{1}{\lambda^2}$

$\mathbb{P}(|X-\mu|\geq\lambda)\leq\frac{\sigma^2}{\lambda^2}$

Problem 2. *(5 points)*

(a) (*2 points*) A group of people consists of 60% of women and 40% of men. The average height of a woman from this group is 160cm, while the average height of a man from this group is 176cm. What is the expected height of a random member of the group in cm?

(b) (*3 points*) Use Chebyshev's inequality to show that for $Z = XY$ where X and Y are two Bernoulli random variables with the same parameter $p = \frac{1}{2}$, we have that

X, Y are ind.

$$\mathbb{P}(Z \geq \frac{3}{4}) \leq \frac{3}{4}.$$

(a) Let X be the height of woman
Let Y be the height of man

$\Rightarrow \mathbb{E}[X] = 160$

$\mathbb{E}[Y] = 176$

$\mathbb{E}[\text{expected height}] = 160 \cdot \frac{60}{100} + 176 \, \frac{40}{100}$

(b) −2.5

Since X, Y are ind and X, Y are Bernoulli random variables

$\Rightarrow \mathbb{E}[Z] = \mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = p \cdot p = p^2 = \frac{1}{4}$

$\mathbb{E}[X] = \frac{1}{2}$, $\mathbb{E}[Y] = \frac{1}{2}$, $var(X) = \frac{1}{4}$, $var(Y) = \frac{1}{4}$

$var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$\Rightarrow \frac{1}{4} = \mathbb{E}[X^2] - \frac{1}{4}$

$var(Y) = \frac{1}{4}$

$\Rightarrow var(XY) = \mathbb{E}[(XY)^2] - (\mathbb{E}[XY])^2$

$\overset{X,Y \text{ are ind.}}{=} \mathbb{E}[X^2]\mathbb{E}[Y^2] - (\frac{1}{4})^2$

$= \cancel{(\frac{1}{4})(\frac{1}{4})} - (\frac{1}{4})^2$

$\frac{1}{2} \cdot \frac{1}{2}$

5.5

Problem 3. *(6 points)*

$Y \backslash X$	0	2	-2
0	0	$\frac{1}{4}$	$\frac{1}{4}$
2	$\frac{1}{4}$	0	0
-2	$\frac{1}{4}$	0	0

For two discrete random variables X and Y, their joint PMF is given by

$$p_{X,Y}(k,l) = \begin{cases} \frac{1}{4}, & \text{if } (k,l) \in \{(0,2),(0,-2),(2,0),(-2,0), \\ 0, & \text{otherwise.} \end{cases}$$

(a) (*2 points*) Compute the marginal PMFs of X and Ym, as well as the conditional PMFs $p_{X|Y}$ and $p_{Y|X}$.

(b) (*2 points*) Compute the expected values $\mathbb{E}[XY]$, $\mathbb{E}[X]$, $\mathbb{E}[Y]$, and the covariance $cov(X,Y)$.

(c) (*2 points*) Are X and Y independent? Justify your answer.

(a) $P_X(k) = \begin{cases} \frac{1}{2} & \text{if } k=0 \\ \frac{1}{4} & \text{if } k=2 \\ \frac{1}{4} & \text{if } k=-2 \\ 0 & \text{o.w.} \end{cases}$ $\qquad P_Y(l) = \begin{cases} \frac{1}{2} & \text{if } l=0 \\ \frac{1}{4} & \text{if } l=2 \\ \frac{1}{4} & \text{if } l=-2 \\ 0 & \text{o.w.} \end{cases}$

$P_{X|Y}(k \mid l=0) = \begin{cases} \frac{1}{4} & \text{if } k=2,-2 \\ 0 & \text{o.w.} \end{cases}$ $\qquad P_{Y|X}(l \mid k=0) = \begin{cases} \frac{1}{4} & \text{if } l=2,-2 \\ 0 & \text{o.w.} \end{cases}$?

$P_{X|Y}(k \mid l=2) = \begin{cases} \frac{1}{4} & \text{if } k=0 \\ 0 & \text{o.w.} \end{cases}$ $\qquad P_{Y|X}(l \mid k=2) = \begin{cases} \frac{1}{4} & \text{if } l=0 \\ 0 & \text{o.w.} \end{cases}$

$P_{X|Y}(k \mid l=-2) = \begin{cases} \frac{1}{4} & \text{if } k=0 \\ 0 & \text{o.w.} \end{cases}$ $\qquad P_{Y|X}(l \mid k=-2) = \begin{cases} \frac{1}{4} & \text{if } l=0 \\ 0 & \text{o.w.} \end{cases}$

(b) Since XY can only be 0 for $(0,2)$, $(0,-2)$, $(2,0)$, $(-2,0)$

$\Rightarrow \mathbb{E}[XY] = 0$

$\mathbb{E}[X] = \sum_{k \in \{0,2,-2\}} k P_X(k) = \frac{1}{2}\cdot 0 + \frac{1}{4}\cdot 2 + \frac{1}{4}(-2) = 0$

$\mathbb{E}[Y] = \sum_{l \in \{0,2,-2\}} l P_Y(l) = \frac{1}{2}\cdot 0 + \frac{1}{4}\cdot 2 + \frac{1}{4}(-2) = 0$

$cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0$

(c) $P_{X,Y}(k,l) \neq P_X(k) P_Y(l)$

for example $P_{X,Y}(0,0) = 0 \neq \frac{1}{2}\cdot\frac{1}{2} = P_X(0) P_Y(0)$

$\Rightarrow$ X, Y are not independent

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Problem 4. *(3 points)*

In a box we have n balls, r of them are red and $n-r$ of them are black (for some given $r \le n$). We pick 2 balls at random. Let X denote the number of black balls among the chosen ones. Compute $\mathbb{E}[X]$ and $var(X)$.

Remark: You have to give a direct argument, not just claim that X is of a certain type.

$$P_X(k) = \begin{cases} \dfrac{\binom{n-r}{k}\binom{r}{2-k}}{\binom{n}{2}} & \text{if } k \in \{0,1,2\} \\ 0 & \text{o.w.} \end{cases}$$

$\Rightarrow$ hypergeometric r.v. since

you pick k balls from $n-r$ black balls

pick $2-k$ balls from r red balls

$\Rightarrow \mathbb{P}(X=k) = \dfrac{\#(k \text{ black balls and } 2-k \text{ red balls})}{\#(\text{pick two ball from } n \text{ balls})}$

$$\mathbb{E}[X] = \sum_{k\in\{0,1,2\}} k\, P_X(k)$$

$$= 1\cdot\frac{\binom{n-r}{1}\binom{r}{1}}{\binom{n}{2}} + 2\cdot\frac{\binom{n-r}{2}\binom{r}{0}}{\binom{n}{2}}$$

$$= \frac{(n-r)(r)}{\binom{n}{2}} + \frac{2\binom{n-r}{2}}{\binom{n}{2}}$$

$$= \frac{(n-r)(r) + \cancel{2}\frac{(n-r)(n-r-1)}{\cancel{2}!\,\cancel{(n-r-2)!}}}{\frac{\cancel{n!}\, n(n-1)}{2!\,\cancel{(n-2)!}}} = \frac{(n-r)(\cancel{n}-1)\cdot 2}{(n)(\cancel{n}-1)} = \frac{2(n-r)}{n}$$

$$var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]$$

$$= \frac{2(n-r)(2n-r-2)}{n(n-1)} - \frac{2(n-r)(n-1)}{n(n-1)} = \frac{2(n-r)(n-r-1)}{n(n-1)}$$

$$\mathbb{E}[X^2] = \sum_{k\in\{0,1,2\}} k^2 P_X(k).$$

$$= 1\cdot\frac{\binom{n-r}{1}\binom{r}{1}}{\binom{n}{2}} + 4\cdot\frac{\binom{n-r}{2}\binom{r}{0}}{\binom{n}{2}}$$

$$= \frac{(n-r)(r) + {}^{2}\cancel{4}\frac{(n-r)(n-r-1)}{\cancel{2}}}{\frac{n!}{2!(n-2)!}}$$

$$= \frac{(n-r)(2n-r-2)\,2}{\frac{n(n-1)}{\cancel{4}}}$$