Midterm I

1) 16

2) 20

3) 20

4) 20

5) 20

96/96

Name: $\underbrace{\text{Chloe Drazen}}_{\text{by writing my name i swear by the honor code}}$

Read all of the following information before starting the exam:

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers, when applicable.
- This test has 5 problems and is worth 100 points.
- Good luck!

1 For four points each, give the following definitions: random variable, compound lottery, expected value, utility function

- Random variable: For a discrete probability space $(\Omega, \mathcal{F}, P)$ and a finite discrete subset $D \in \mathbb{R}$, a random variable X is defined $X : \Omega \rightarrow D$.

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- Compound lottery: (a lottery of lotteries) For a set $\mathcal{L}$ of lotteries $L_1, L_2, \dots, L_n$ that each have probability $q_1, q_2, \dots, q_n$ of being chosen. Written $\langle (L_1, q_1), (L_2, q_2), \dots, (L_n, q_n) \rangle$

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- Expected value: for discrete probability space $(\Omega, \mathcal{F}, P)$ and random variable $X : \Omega \rightarrow D$, the expected value is defined $E(X) = \sum_{x \in D} x \cdot P(x)$

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- Utility function: For a set of prizes $L_1, \dots, L_n$ that have preference ordering $L_1 \succeq \dots \succeq L_n$ over a lottery, a utility function is a preference preserving function $u : \mathcal{F} \rightarrow [0,1]$ such that:

$$u(L_1) \geq u(L_2) \text{ iff } L_1 \succeq L_2$$

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we define $u(L_1) = 1$
$u(L_n) = 0$

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2 Assume you are playing a die rolling game, where you roll a six sided die. What is the probability of rolling $E = \{1, 4\}$.

$$\frac{2}{6} = \boxed{\frac{1}{3}}$$

$\Omega = \{1, 2, 3, 4, 5, 6\}$

Assuming it is a fair die, the probability of rolling any $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, or $\{6\}$ is equal, and the probability of rolling anything else is 0.

Therefore, the probability of rolling any # is $\frac{1}{6}$.

Let $P(1)$ denote prob of rolling 1

$P(4)$ denote prob of rolling 4

E = roll a 1

F = roll a 4

$P(1) \cap P(4) = \emptyset$ Therefore these event are disjoint

By our def of d.p.s if $E \cap F = \emptyset$, then

$P(E \cup F) = P(E) + P(F)$

$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

3 Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a discrete probability space. Prove that $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

let's define E' = all elements x s.t. $x \in E$, but $x \notin E \cap F$
F' = all elements y s.t. $y \in F$, but $y \notin E \cap F$

① $E' \cap F' = \emptyset$
② $E' \cap (E \cap F) = \emptyset$
③ $F' \cap (E \cap F) = \emptyset$

④ $E' \cup (E \cap F) = E$
⑤ $F' \cup (E \cap F) = F$

$P(E \cup F) = P(E' \cup F' \cup (E \cap F))$
by ①, ②, & ③ above, these three event are mutually exclusive, and by the definition of d.p.s if $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.
$P(E' \cup F' \cup (E \cap F)) = P(E') + P(F') + P(E \cap F)$

we also know $P(E) = P(E') + P(E \cap F) \Rightarrow P(E') = P(E) - P(E \cap F)$
and $P(F) = P(F') + P(E \cap F) \Rightarrow P(F') = P(F) - P(E \cap F)$
by same reason stated above in def of d.p.s.

$P(E') + P(F') + P(E \cap F) = P(E) - P(E \cap F) + P(F) - \cancel{P(E \cap F)} + \cancel{P(E \cap F)}$
$P(E \cup F) \qquad = P(E) + P(F) - P(E \cap F)$ ✓ :)

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4 Let $\sim$ be the relation on the set of real numbers defined by $x \sim y$ iff $x - y$ is an integer. a) Prove that this is an equivalence relation. b) Prove that if $x \sim x'$ and $y \sim y'$, then $(x+y) \sim (x'+y')$.

a) In order to be an equivalence relation, it needs to be reflexive, transitive, and symmetric:

Reflexive: Prove $x \sim x$: $x \sim x = x - x = 0$
$0 \in$ integers, so this is reflexive.

Symmetric: Prove if $x \sim y$, then $y \sim x$:
let's say $x - y = a$, where $a \in$ integers.
$y - x = -1(x-y) = \underset{\text{integer}}{-1} \cdot a = -a$. Because multiplying an integer by another integer is still an integer, $-a \in$ integers, so $y \sim x$
and this is symmetric.

Transitive: Prove if $x \sim y$ and $y \sim z$, then $x \sim z$:
let's say $x - y = a \in$ integers
$y - z = b \in$ integers $\Rightarrow z = y - b$
Then $x - z = x - (y-b) = x - y - b = a - b = c$
and since subtracting one integer (b) from another (a) is still an integer, $c \in$ integers, and $x \sim z$ and this is transitive.

Therefore, $\sim$ is an equivalence relation. :) ✓

b) Prove that if $x \sim x'$ and $y \sim y'$ then $(x+y) \sim (x'+y')$
let's define $x - x' = a \in$ integers
$y - y' = b \in$ integers

If we add together $a + b = c$ because addition of 2 integers is an integer, $c \in$ Integer.
$a + b = c = (x - x') + (y - y') = (x+y) - (x'+y') \in$ Integers.
This implies $(x+y) \sim (x'+y')$:) ✓

5 You are on *The Price is Right!* You are going to play a game called *Temptation.* In this game you are offered 3 prizes and given their dollar values. From these dollar values you must construct the price of a car. Once you are shown all the prizes (and constructed your guess for the price of a car) you must make a choice between taking all the prizes and leaving or hoping that you have chosen the right numbers in the price of the car. Construct all relevant lotteries for this game.

There are 3 possible prizes to be won: A_0, A_1, A_2

A_0 = win nothing

A_1 = win 3 prizes

A_2 = win car

There are 2 possible lotteries to choose from

↳ 1 where you choose to take 3 prizes (L_1)

↳ 1 where you guess prize of car (L_2)

↳ Either win nothing

↳ or win car

We will define L_1 and L_2 as follows:

$$L_1 = \langle (A_0, p_0 = 0), (A_1, p_1 = 1), (A_2, p_2 = 0) \rangle$$

$$L_2 = \langle (A_0, p_0 = q), (A_1, p_1 = 0), (A_2, p_2 = 1-q) \rangle$$

where q is the probability $[0,1]$ that you correctly guess the price of the car in order to win.