1. (a) Calculate the MM strategies for both players and their safety levels and (b) find a Nash equilibrium for the bimatrix game

$$(A, B) = \begin{bmatrix} (1, -1) & (3, -2) \\ (2, -4) & (-1, 2) \end{bmatrix}$$

[10 points] (a)

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \qquad B^T = \begin{bmatrix} -1 & -4 \\ -2 & 2 \end{bmatrix}$$

$$p^* = \left(\frac{-3}{-5} = \frac{3}{5}, \frac{2}{5}\right) \qquad q^* = (4/7, 3/7)$$

$$V_I = \frac{-7}{-5} = \frac{7}{5} \qquad V_{II} = -\frac{10}{7}$$

[10 points] (b)

equalizing $\left((6/7, 1/7), \left(\frac{-4}{-5} = \frac{4}{5}, \frac{1}{5}\right)\right)$

2. Given that the inverse of the payoff matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

is

$$A^{-1} = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix}$$

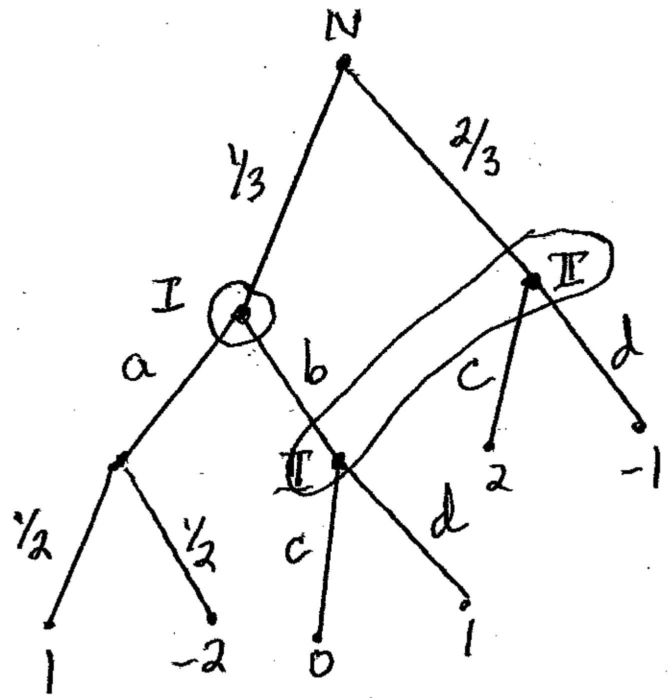
calculate the optimal strategies for both players and the value of this matrix game.

$$v(A) = \left[\frac{1}{3}(5+4+3)\right]^{-1} = \left[\frac{11}{3}\right]^{-1} = \frac{3}{11}$$

$$p^* = (5/3, 4/3, 2/3)\left(\frac{3}{11}\right) = (5/11, 4/11, 2/11)$$

$$q^* = (2/3, 4/3, 5/3)\left(\frac{3}{11}\right) = (2/11, 4/11, 5/11)$$

3. Calculate the matrix that is the strategic form of the game given in extensive form below.



$$\begin{array}{c|cc} & c & d \\ \hline \begin{array}{c} a \\ b \end{array} & \multicolumn{2}{c}{\begin{bmatrix} 7/6 & -5/6 \\ 4/3 & -1/3 \end{bmatrix}} \end{array}$$

(ac) $\left(\frac{1}{3}\right)\left[\left(\frac{1}{2}\right)(1) + \left(\frac{1}{2}\right)(-2)\right] + \left(\frac{2}{3}\right)(2) = \left(\frac{1}{3}\right)\left(-\frac{1}{2}\right) + \frac{4}{3} = \frac{7}{6}$

(ad) $\left(\frac{1}{3}\right)\left[\left(\frac{1}{2}\right)(1) + \left(\frac{1}{2}\right)(-2)\right] + \left(\frac{2}{3}\right)(-1) = -\frac{1}{6} - \frac{4}{6} = -\frac{5}{6}$

(bc) $\left(\frac{1}{3}\right)(0) + \left(\frac{2}{3}\right)(2) = 4/3$

(bd) $\left(\frac{1}{3}\right)(1) + \left(\frac{2}{3}\right)(-1) = -1/3$

4. The price function for companies #1 and #2 who produce quantities q_1 and q_2 of a product is $P(q_1, q_2) = 110 - q_1 - q_2$. The production costs to the companies are $20q_1 + 30$ and $30q_2 + 9$ respectively. (a) Show that if company #1 produces q_1 units, then the optimal production level for company #2 is $q_2 = 40 - \frac{q_1}{2}$. (b) Calculate the value of q_1 such that if company #2 produces $q_2 = 40 - \frac{q_1}{2}$ units, then its profit will be zero.

[7 points] (a) $U_2(q_1, q_2) = (110 - q_1 - q_2)q_2 - 30q_2 - 9$

$= 110q_2 - q_1q_2 - q_2^2 - 30q_2 - 9$

$\frac{\partial U_2}{\partial q_2}(q_1, q_2) = 110 - q_1 - 2q_2 - 30 = 0$

$2q_2 = 80 - q_1 \quad q_2 = 40 - \frac{q_1}{2}$

[13 points] (b) $U_2(q_1, 40 - q_1/2) = (110 - q_1 - 40 + \frac{q_1}{2})(40 - \frac{q_1}{2})$
$- 30(40 - q_1/2) - 9 = 0$

$(70 - \frac{q_1}{2})(40 - \frac{q_1}{2}) - 30(40 - \frac{q_1}{2}) = 9$

$(40 - \frac{q_1}{2})^2 = 9 \quad 40 - \frac{q_1}{2} = \pm 3$

$\frac{q_1}{2} = 37$ or $43 \quad q_1 = 74$ or 86

but if $q_1 = 86$ then $q_2 = -3$ so $q_1 = 74$

5. Prove that a symmetric matrix game is fair, that is, its value is zero.

Since the game is symmetric then $A^T = -A$ so

$p^T A p = (p^T A p)^T = p^T A^T p = p^T(-A)p = -p^T A p$

so $p^T A p = 0$.

If $V = V(A) > 0$ then by the Minimax Theorem there exists p^* such that $p^{*T} A q \geq V > 0$ for all q. But $p^{*T} A q = 0$ if $q = p^*$ so $V \leq 0$.

If $V < 0$ then there exists q^* such that $p^T A q^* \leq V < 0$ for all p. But $p^T A q^* = 0$ if $p = q^*$ so $V = 0$.