Math 167-1, Winter 2021
Mathematical Game Theory
Final Exam

Instructions:

1. The test is conducted online through **Gradescope**. You have **24 hours, Wed Mar 17 08:00 AM (PT) to Thu Mar 18 08:00 AM (PT)** to complete and submit the test in **Gradescope**. There are **five questions** worth a total of **50 points**.

2. For full credit, show all of your work legibly and always justify your answers.

3. The test is an open book, notes, and the internet. However, the usage of these resources must be conducted according to *Academic Honesty Principles.* In particular, **collaborations are not allowed**, and the submission must be your individual work, just as it would be the case with an in-person exam. **Posting parts of the test or their solutions anywhere and seek or provide assistance is not allowed.**

4. Together with the test, everyone must sign and submit the following statement:

   "I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation."

5. Everyone must comply with the rules above and other principles of the Student Conduct Code https://www.deanofstudents.ucla.edu (see in particular Section 102.01 on academic dishonesty). Deviation from the rules may render tests void.

Academic Honesty Statement. Please sign and submit the statement below.

"I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation."

Question 1. *10pts.*

Suppose that Alice and Bob play a two-round game. Each round is a zero-sum game that does not affect the other round. Prove that the value of the game is the sum of the values in each round.

Question 2. *10pts.*

Consider a k-player general sum game with finite strategy spaces $\{S_i\}_{i\in[k]}$ and payoff functions $\{u_i\}_{i\in[k]}$ for player. Assume that there exists $f : S_1 \times S_2 \times \cdots \times S_k \to \mathbb{R}$ such that $f(s_i, \mathbf{s}_{-i}) < f(\hat{s}_i, \mathbf{s}_{-i})$ if and only if $u_i(s_i, \mathbf{s}_{-i}) > u_i(\hat{s}_i, \mathbf{s}_{-i})$ for all $s_i, \hat{s}_i, \mathbf{s}_{-i}$. Does this game have a pure Nash equilibrium? Justify your answer.

Question 3. *10pts.*

A marketing company has 100 sales representatives that have to advertise a new product to 100 potential customers. Initially, the sales representatives did not coordinate their calls and ended up placing 25 calls each. A post-advertisement survey revealed that the customers were unhappy with the advertisement campaign because they received 25 calls each. Can the marketing company manager reduce the calls' volume while maintaining the same level of outreach for the new product? Justify your answer.

Note: Sales representatives can reach customers only from their initial list of 25.

Question 4. *10pts.*

Consider n students and n colleges with preference profiles given by a compatibility matrix $A = (a_{ij})$. Anna and Yuval play a *find-a-better-match* game as follows. Anna names a student s_1, and Yuval names a college c_1. Next, Anna names a student s_2 that has a higher compatibility score with c_1 than s_1. Afterward, Yuval names a college c_2 that is more compatible with s_2 than c_1, and so on. The player that can no longer name a college or student loses. Does any of the players have a winning strategy? Justify your answer.

Question 5. *10pts.*

- Discuss the similarities and differences between the core and Shapley value.
- Discuss the similarities and differences between the Shapley value and Nash bargaining solution.

21W-MATH167-1 Final

DAVID DAVINI

TOTAL POINTS

48 / 50

QUESTION 1

1 10 / 10

✓ **+ 10 pts** Full credit - need to use some formal definition of the "value" of a game for full credit here.

For instance: call the players $$A$$ and $$B$$, and suppose that player $$A$$'s safety strategies are $$x_1$$ and $$x_2$$ for the first and second rounds respectively. Let the matrices of the two games be $$R_1$$ and $$R_2$$ respectively.

That means $$x_1 R_1 y_1 \geq v_1$$ and $$x_2 R_2 y_2 \geq v_2$$ for any strategies $$y_1, y_2$$ of player $$B$$.

So, if player $$A$$ adopts the strategy of $$x_1$$ then $$x_2$$, their payoff is at least

$$\mathbb{E}[x_1 R_1 y_1 + x_2 R_2 Y_2] \geq v_1 + v_2$$

no matter what player $$B$$ chooses, even though $$Y_2$$ may depend on what happened in round 1. (that's why it's written as a random variable - the inequality still holds though since $$\mathbb{E}[x_2R_2 Y_2]$$ is a weighted average). Likewise, player $$B$$ can guarantee a payoff of at least $$-(v_1 + v_2)$$ by an analogous strategy.

+ 7 pts Says that the best strategy is to play the successive best strategies without really getting mathematically into why.

Yes, this is true but it is what you are being asked to prove. Note that it isn't enough just to show that playing the best strategy both times gives an expected payoff of $$v_1 + v_2$$; we also need to show that this is safety (or similar).

QUESTION 2

2 10 / 10

✓ **- 0 pts** Correct

QUESTION 3

3 10 / 10

✓ **+ 10 pts** Full credit; the intended solution is that this describes a 25-regular bipartite graph. Thus, by problem 3.2 from Homework 4 [a direct consequence of Hall's marriage theorem] there must be a perfect matching, which is the best possible solution.

QUESTION 4

4 10 / 10

✓ **+ 10 pts** Full credit: Yuval has a winning strategy.

By the theorem in class (lecture 3/3/21) - there exists a unique stable matching $$M$$ between the set of students and the set of colleges. Yuval can adopt the strategy of, whenever Anna plays a student $$s$$, to play the college $$M(s)$$.

To show that this is a winning strategy, we have to show two things:

(1) that this is always a legal play. Suppose that the game so far has gone $$s_1, M(s_1), s_2, M(s_2), \dots, s_k$$.

Then, for Yuval's play to be legal, we need student $$k$$ to prefer $$M(s_k)$$ to $$M(s_{k - 1})$$. But we know from the fact that Anna was allowed to play $$s_k$$ that college $$M(s_{k - 1})$$ prefers student $$s_k$$ to student $$s_{k - 1}$$. So if student $$s_k$$ preferred $$M(s_{k - 1})$$, then this would be an instability (as $$s_k$$ and $$M(s_{ k - 1})$$ would have incentive to defect). That means Yuval's play is always legal.

(2) that the game terminates. But by the structure of the game, the compatibility $$a_{t}$$ always strictly increases; since it only has a finite number of possible values, the game must terminate.

So Yuval has a winning strategy.

[Note: if we don't assume that all the values are distinct, then neither (1) nor (2) hold and there may be a winning strategy for Anna].

+ **8 pts** Names the correct strategy, and shows (or effectively shows) only part (1) of the above; that Yuval cannot lose.

Note that we also need to justify that Yuval eventually does win (that is, that the players don't end up in an endless loop). There is no assumption that players are not allowed to repeat students or colleges: Piazza @29_f2

Of course, the structure of the game makes it _impossible_ to repeat student / college pairs anyway, which is why this is not an assumption!

+ **4 pts** Suggests the strategy in which Yuval simply names the largest entry in each row (i.e. the college that is most preferred by the named student.) This does not result in a win for Yuval. Suppose the preference matrix is as follows:

$$ \begin{bmatrix}1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Then if Anna picks student 1, the strategy calls for Yuval to respond with college 3; but then Anna can pick student 3 and win. (In fact, Yuval's winning strategy here is to respond with college 1).

There's no assumption that the ranks are 1 through $$n$$ in each row. See the lecture from March 3.

QUESTION 5

5 **8 / 10**

✓ **- 0 pts** Correct

- 2 **Point adjustment**

1. A principal difference between a Shapley value and the core is that the latter is defined for a single instance of a cooperative game whereas the former is defined as a mapping from the set of characteristic functions to the set of shares. In particular, cores of two different characteristic functions are independent of one another whereas the Shapley values are not: see the Additivity axiom.
2. Similarly, both the Shapley value and Nash bargaining solution are not merely instances of fair shares or values for the players -- they are functions from the space of instances of cooperative games to the set of shares. Again, solutions for various configurations are connected to one another: see Independence of Irrelevant Alternatives axiom for Nash bargaining.

QUESTION 6

6 Academic honesty statement **0 / 0**

✓ **- 0 pts** Correct

1. • Since each round does not affect the other, we can assume the players decide their strategies for both rounds before playing the first round (since the outcome of round 1 cannot change the optimal strategy in round 2)

• Let G_1 and G_2 be the zero-sum games for round 1 & 2 resp.
• Let A_1 and A_2 be the payoff matrices for G_1 & G_2 resp.
• Let $\Delta_{m_1}, \Delta_{n_1}$ and $\Delta_{m_2}, \Delta_{n_2}$ be the sets of mixed strategies for each player for G_1 and G_2 resp.

• Denote $V_1(x,y) = x^T A_1 y \quad \forall x \in \Delta_{m_1}, y \in \Delta_{n_1}$
$V_2(x,y) = x^T A_2 y \quad \forall x \in \Delta_{m_2}, y \in \Delta_{n_2}$

• Let $V_1 = \max_{x \in \Delta_{m_1}} \min_{y \in \Delta_{n_1}} V(x,y)$ and $V_2 = \max_{x \in \Delta_{m_2}} \min_{y \in \Delta_{n_2}} V(x,y)$ the values of G_1 & G_2

• Let G be the two-round game consisting of G_1 and G_2
• Note G has mixed strategy sets $\Delta_{m_1} \times \Delta_{m_2}$ and $\Delta_{n_1} \times \Delta_{n_2}$

• Also, $V(x,y)$ = "payoff when player 1 plays x and player 2 plays y"
$= x_1^T A_1 y_1 + x_2^T A_2 y_2$

where $x = (x_1, x_2) \in \Delta_{m_1} \times \Delta_{m_2}$ and $y = (y_1, y_2) \in \Delta_{n_1} \times \Delta_{n_2}$
• Let V be the value of G.

• For convenience, we leave the set we maximize over implied, eg. $\max_{x \in \Delta_{m_1} \times \Delta_{m_2}}$ is $\max_x$

• Note $V = \max_x \min_y V(x,y) = \max_{x_1,x_2} \min_{y_1,y_2} (V_1(x_1,y_1) + V_2(x_2,y_2))$

$= \max_{x_1,x_2} \min_{y_1,y_2} V_1(x_1,y_1) + \max_{x_1,x_2} \min_{y_1,y_2} V_2(x_2,y_2)$ (property of mins/maxes)

$= \max_{x_1} \min_{y_1} V_1(x_1,y_1) + \max$

1 (cont) • We will use a property of min's, namely that

$$\min_{\substack{a\in A\\ b\in B}} \left(f(a)+g(b)\right) = \min_{a\in A} f(a) + \min_{b\in B} g(b) \qquad (\star)$$

for any sets A, B and functions $f: A\to\mathbb{R}$ $g: B\to\mathbb{R}$

• (an equivalent property for max's exists)

• So then $V = \max_{x} \min_{y} V(x,y) = \max_{x_1, x_2} \min_{y_1 y_2} \left(V_1(x_1,y_1) + V_2(x_2,y_2)\right)$

$$= \max_{x_1,x_2} \left(\min_{y_1} V_1(x_1,y_1) + \min_{y_2} V_2(x_2,y_2)\right) \qquad \text{by } (\star)$$

$$= \max_{x_1} \min_{y_1} V_1(x_1,y_1) + \max_{x_2} \min_{y_2} V_2(x_2,y_2) \qquad \text{by } (\star)$$

$$= V_1 + V_2$$

□

1 **10 / 10**

✓ **+ 10 pts** Full credit - need to use some formal definition of the "value" of a game for full credit here.

For instance: call the players A and B, and suppose that player A's safety strategies are x_1 and x_2 for the first and second rounds respectively. Let the matrices of the two games be R_1 and R_2 respectively.

That means $x_1 R_1 y_1 \geq v_1$ and $x_2 R_2 y_2 \geq v_2$ for any strategies y_1, y_2 of player B.

So, if player A adopts the strategy of x_1 then x_2, their payoff is at least

$$\mathbb{E}[x_1 R_1 y_1 + x_2 R_2 Y_2] \geq v_1 + v_2$$

no matter what player B chooses, even though Y_2 may depend on what happened in round 1. (that's why it's written as a random variable - the inequality still holds though since $\mathbb{E}[x_2R_2 Y_2]$ is a weighted average). Likewise, player B can guarantee a payoff of at least $-(v_1 + v_2)$ by an analogous strategy.

+ 7 pts Says that the best strategy is to play the successive best strategies without really getting mathematically into why.

Yes, this is true but it is what you are being asked to prove. Note that it isn't enough just to show that playing the best strategy both times gives an expected payoff of $v_1 + v_2$; we also need to show that this is safety (or similar).

2. • **Yes** the game has a pure Nash equilibrium.

• The proof is very similar to the proof of pure NE in the case of a potential function game.

: Pf: • Let $s^* \in S_1 \times S_2 \times \ldots S_k$ be a minimizer of f, that is $\min_s f(s) = f(s^*)$

• (We know such an s^* exists because $S_1 \times \ldots \times S_k$ is finite)

• We claim s^* is a pure NE of the game.

• Let $\hat{s} = (\hat{s}_i, s^*_{-i})$ for some $\hat{s}_i \in S_i$, for some $i \in [k]$

(that is, $\hat{s}$ is the strategy profile that occurs when player i deviates w/ strategy $\hat{s}_i$)

• We aim to show $u_i(\hat{s}) \leq u_i(s^*)$

• Note $f(\hat{s}) \geq f(s^*)$ by def. of s^*.

• Case 1: $f(\hat{s}) > f(s^*)$

• Then $u_i(s^*) > u_i(\hat{s})$ (by problem assumption)

• Case 2: $f(\hat{s}) = f(s^*)$

• Suppose $u_i(\hat{s}) > u_i(s^*)$

• Then $f(\hat{s}) < f(s^*)$, contradiction

• Suppose $u_i(\hat{s}) < u_i(s^*)$

• Then $f(\hat{s}) > f(s^*)$, contradiction

• Thus $u_i(\hat{s}) = u_i(s^*)$

• This shows $u_i(\hat{s}) \leq u_i(s^*)$ $\forall \hat{s} = (\hat{s}_i, s^*_{-i})$, so s^* is a pure NE. $\square$

2 10 / 10

✓ **- 0 pts** Correct

Yes It's possible to reduce call volume to 100 calls total.

3. • Let $S = \{\text{sales representatives}\}$ and $C = \{\text{potential customers}\}$

• Let G be the graph representing coordination, that is, each edge $(s, c) \in E \subseteq S \times C$ represents that representative s called customer c. Denote $n = |S| = |C| = 100$

(Also, denote $V = S \cup C$.)

• Note G is bipartite.

• Also, note G is k-regular, where $k = 25$

• Claim: • G has a perfect matching

Pf: • By Hall's Marriage Thm, suffices to show that every $S' \subseteq S$ satisfies $|S'| \leq |f(S')|$ where f is the function $f(x) = \{y \in C : (x, y) \in E\}$ $\forall x \subseteq S$

• Denote $E(X) = \{(x,y) \in E : x \in X \text{ or } y \in X\}$ $\forall X \subseteq V$

• Let $S' \subseteq S$

• Since G is k-regular, $|E(S')| = k|S'|$

• Also, $|E(f(S'))| = k|f(S')|$

• But $E(S') \subset E(f(S'))$ (by def of $E(\cdot)$ and $f(\cdot)$)

• So $|E(S')| \leq |E(f(S'))|$

$\Rightarrow k|S'| \leq k|f(S')|$

$\Rightarrow |S'| \leq |f(S')|$ □

• So G has a perfect matching M.

• A perfect matching contains every vertex but has no adjacent edges.

• Thus, if every sales representative s calls their match c in M, every customer will be called exactly once.

• Hence call volume will be reduced to $n = 100$ calls, with same outreach of $n = 100$

3 **10 / 10**

✓ **+ 10 pts** Full credit; the intended solution is that this describes a 25-regular bipartite graph. Thus, by problem 3.2 from Homework 4 [a direct consequence of Hall's marriage theorem] there must be a perfect matching, which is the best possible solution.

4. **Yes** Player 2 has a winning strategy (Yuval)

- <u>Winning Strategy for Player 2</u>: - For every s_i Player 1 plays, Player 2 should play c_j that is matched with s_i in the unique stable matching M

- Note there is a <u>unique</u> stable matching M since the preference profiles are given by a compatibility matrix $A=(a_{ij})$

- We will prove this strategy is indeed a winning strategy.

- <u>Claim</u>: - Player 2 can <u>always</u> play according to this strategy, (provided they never deviate from the strategy)

  <u>Pf</u>: - In player 2's <u>first move</u> they can pick any college so clearly they can pick the c_j matched to Player 1's s_i by M.

  - Otherwise, let s_h, s_i be the last two moves played by Player 1 and c_j be the last move played by Player 2, and let it be Player 2's turn.

  - Denote c_k as the college matched to s_i by M.

  - Since Player 2 last played c_j, then c_j must be matched to s_h by M.

  - Also, since M is a stable matching, there are no instabilities.

  - Then since c_j prefers s_i to s_h, s_i must prefer c_k to c_j (otherwise (c_j, s_i) would be an instability)
  - Thus Player 2 can play c_k, and so follow the strategy □

- Claim: - The game cannot go on forever.

  Pf: - Suppose the game did go on forever. Let (s_1, c_1) (s_2, c_2)... be the pairs played in every 2 rounds.

  - Note since there are only n students and n colleges, there are only n^2 pairs, so at some point the game returns to a previous pair, that is $(s_1, c_1) = (s_i, c_i)$ for some $i \in \mathbb{N}$

  - We know $i \neq 1$, since Player 1 can't pick the same student twice in a row

  - Then from the rules of the game we know
    $$a_{11} < a_{21} < a_{22} < a_{32} < \dots < a_{ii} = a_{11}$$

    (since each time a more compatable is selected)

  - Then $a_{11} < a_{11}$, contradiction ↯

- Since Player 2 can always play according to our strategy and the game must end, then we know Player 1 must lose the game when Player 2 plays our strategy.

- Thus our strategy is in fact a winning strategy.

- Since Player 2 has a winning strategy, Player 1 has no winning strategy.

4 **10 / 10**

✓ **+ 10 pts** Full credit: Yuval has a winning strategy.

By the theorem in class (lecture 3/3/21) - there exists a unique stable matching $$M$$ between the set of students and the set of colleges. Yuval can adopt the strategy of, whenever Anna plays a student $$s$$, to play the college $$M(s)$$.

To show that this is a winning strategy, we have to show two things:

(1) that this is always a legal play. Suppose that the game so far has gone $$s_1, M(s_1), s_2, M(s_2), \dots, s_k$$.

Then, for Yuval's play to be legal, we need student $$k$$ to prefer $$M(s_k)$$ to $$M(s_{k - 1})$$. But we know from the fact that Anna was allowed to play $$s_k$$ that college $$M(s_{k - 1})$$ prefers student $$s_k$$ to student $$s_{k - 1}$$. So if student $$s_k$$ preferred $$M(s_{k - 1})$$, then this would be an instability (as $$s_k$$ and $$M(s_{ k - 1})$$ would have incentive to defect). That means Yuval's play is always legal.

(2) that the game terminates. But by the structure of the game, the compatibility $$a_{t}$$ always strictly increases; since it only has a finite number of possible values, the game must terminate.

So Yuval has a winning strategy.

[Note: if we don't assume that all the values are distinct, then neither (1) nor (2) hold and there may be a winning strategy for Anna].

+ 8 pts Names the correct strategy, and shows (or effectively shows) only part (1) of the above; that Yuval cannot lose.

Note that we also need to justify that Yuval eventually does win (that is, that the players don't end up in an endless loop). There is no assumption that players are not allowed to repeat students or colleges: Piazza @29_f2

Of course, the structure of the game makes it _impossible_ to repeat student / college pairs anyway, which is why this is not an assumption!

+ 4 pts Suggests the strategy in which Yuval simply names the largest entry in each row (i.e. the college that is most preferred by the named student.) This does not result in a win for Yuval. Suppose the preference matrix is as follows:

$$ \begin{bmatrix}1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Then if Anna picks student 1, the strategy calls for Yuval to respond with college 3; but then Anna can pick student 3 and win. (In fact, Yuval's winning strategy here is to respond with college 1).

There's no assumption that the ranks are 1 through $$n$$ in each row. See the lecture from March 3.

- Let G be a cooperative game w/ transferable utilities, defined by characteristic func. $V: 2^S \to \mathbb{R}$

5. - The core of G is the set of allocation vectors ψ s.t.

(1) $\sum_{i \in [n]} \psi_i = V([n])$ (efficiency)

(2) $\sum_{i \in S} \psi_i \geq V(S) \quad \forall S \subseteq [n]$ (stability)

- The shapley value function Ψ maps characteristic functions v to allocation vectors $\Psi(v)$, s.t.

  - for any characteristic function v,

    (1) $\Psi(v)$ satisfies efficiency (stated above)

    (2) $\psi_i(v) = \psi_j(v) \quad \forall i, j$ s.t. $v(S \cup \{i\}) = v(S \cup \{j\}) \quad \forall S \not\ni i, j$ (symmetry)

    (3) $\psi_i(v) = 0 \quad \forall i$ s.t. $v(S \cup \{i\}) = v(S) \quad \forall S \subseteq [n]$

  - for any characteristic functions v, u,

    (4) $\Psi(v+u) = \Psi(v) + \Psi(u)$ (additivity)

- Similarities between core and shapley value: (for a char. func. v)

1. - Both the shapley value and an element of the core satisfy efficiency.

2. - Both the shapley value and an element of the core are allocation vectors.

S(cont)

· Differences between core and shapley values: (of a char. func. V):

1. · The core can have multiple elements, whereas the shapley value is unique
2. · The core can have no elements, whereas the shapley value always exists
3. · A core element can be asymmetrical, whereas the shapley value is symmetric
4. · A core element must satisfy stability, whereas the shapley value need not.

· The Nash bargaining solution F^N maps bargaining problems (S,d) to agreement points $F^N(S,d) \in S$ satisfying

(1) F^N is affine covariant $\left(F^N(\psi(S), \psi(d)) = \psi(F^N(S,d)) \right)$ for any $\alpha_1, \alpha_2, \beta_1, \beta_2$ w/ $\alpha_1, \alpha_2 > 0$)

(2) F^N is pareto optimal ($F^N(S,d) = a$ and $a' \geq a$ implies $a = a'$, for any $a' \in S$)

(3) F^N is Symmetric (If $(x,y) \in S \Leftrightarrow (y,x) \in S$ and $d_1 = d_2$ then $F^N(S,d) = (a,a)$ for some $(a,a) \in S$)

(4) F^N is Independent of Irrelevant Alternatives ($\forall$ (S,d) and (S',d) s.t. $S \subseteq S'$ and $F(S',d) \in S$, $F(S,d) = F(S',d)$)

5 (cont) · Similarities between shapley values and Nash bargaining solutions:

1. · Both always exist: · every char. func. v has a shapley value $\psi(v)$
every bargaining problem (S, d) has a Nash bargaining sol'n $F^N(S, d)$

2. · Both are always unique: · every char. func v has unique shapley value $\psi(v)$
· every bargaining problem (S, d) has unique Nash bargaining sol'n $F^N(S, d)$

3. · Both have explicit solutions: · $F^N(S, d) = a$ is the maximizer of $(x_1 - d_1)(x_2 - d_2)$
subject to $x_1 \geq d_1$ and $x_2 \geq d_2$ $(x_1, x_2) \in S$
· $\psi(v) = \underset{\pi}{\mathbb{E}}(\varphi_i(\pi, v))$ where $\varphi_i(\pi, v) = v(\pi[k]) - v(\pi[k-1])$
and π permutation

4. · Both represent fair solutions
5. · Both are symmetric in their respective ways

· Differences between shapley values and Nash bargaining solutions:

1. · Shapley value is an allocation vector from a cooperative game w/ transferable utilities whereas Nash bargaining solution is a possible payoff from a bargaining problem

2. · Shapley values are efficient, whereas Nash bargaining sol'ns need not be.

5 8 / 10

✓ **- 0 pts** **Correct**

- 2 **Point adjustment**

1. A principal difference between a Shapley value and the core is that the latter is defined for a single instance of a cooperative game whereas the former is defined as a mapping from the set of characteristic functions to the set of shares. In particular, cores of two different characteristic functions are independent of one another whereas the Shapley values are not: see the Additivity axiom.
2. Similarly, both the Shapley value and Nash bargaining solution are not merely instances of fair shares or values for the players -- they are functions from the space of instances of cooperative games to the set of shares. Again, solutions for various configurations are connected to one another: see Independence of Irrelevant Alternatives axiom for Nash bargaining.