2021 Spring Math 151B
Applied Numerical Methods

Midterm

Instructions: You will have 24 hours (Pacific Time: May 6, 1 pm - May 7, 1 pm) to complete the exam. **You must submit your solutions before May 7, 1 pm via gradescope, late submission will not be accepted.**

Notice that you are permitted to use notes, texts, and computers for Matlab or Python coding during the exam. **But you are not permitted to use human resources (including Chegg, Math Stack Exchange, etc.) or to collaborate. Violations of these rules will be regarded as academic dishonesty and will be reported to the office of the Dean of Students.**

You must **show your work** to receive credit(if you only show your final results, you will get at most half of the full points). Please circle or box your final answers. Please Organize your work, in a reasonably neat and coherent way, otherwise your solutions may not be graded.

Name: ______________________________
Student ID number: ______________________
Section: _____________________________

"I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation." Please read the statement and sign here:

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(This statement must be signed by the student. If it is not signed, the evaluation must be given a failing grade.)

Question	Points	Score
1	14	
2	11	
3	15	
4	10	
Total:	50	

Problem 1.

Consider solving the autonomous differential equation

$$\frac{dy}{dt} = f(y), \quad t \in [t_0, T], \quad y(t_0) = y_0$$

with the following backward Euler's method:

$$y_{i+1} = y_i + hf(y_{i+1}).$$

Assume that f is smooth.

(a) [7pts.] Derive the leading term of the local truncation error of this numerical method.

(b) [7pts.] Determine the interval of absolute stability for this method. (Please show your work.)

Problem 2.

The initial value problem

$$y' = y - y^2, \quad t \in [0, 2],\ y(0) = 2 \tag{1}$$

has analytic solution $y(t) = \frac{2}{2-e^{-t}}$.

(a) [7pts.] Assume the Euler's method is used to solve the above initial value problem (1). The interval of absolute stability for the Euler's method is $(-2, 0)$. What is stability restrictions(i.e., the estimated results have qualitative agreement with the analytic solution) on step size h using Euler's method?

(b) [4pts.] If we use the Trapezoidal method, whose interval of absolute stability is $(-\infty, 0)$, to solve this problem, what is stability restrictions on step size h using Trapezoidal method?

Problem 3.

The higher order differential equation and initial conditions are shown as follows:

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = t^2 + \cos(t), \quad y(0) = 0.5, \quad y'(0) = 1.$$

(a) [6pts.] Transform the above initial value problem into an equivalent first order differential system, including initial conditions.

(b) [4pts.] Express the system in (a) in matrix form, write the initial condition in vector form.

(c) [5pts.] Using the second order Runge-Kutta method as follows

$$\begin{aligned} \vec{y}* &= \vec{y_i} + h\vec{F}(t_i, \vec{y_i}) \\ \vec{y}_{i+1} &= \vec{y_i} + \frac{h}{2}(\vec{F}(t_i, \vec{y_i}) + \vec{F}(t_{i+1}, \vec{y*})). \end{aligned}$$

to solve system in (b) with step size h, what is the matrix form of the iteration formula? (You do not need to combine the above two equations into one equation)

Problem 4.

Let α be a constant and consider the numerical method

$$y_{k+1} = y_k + hf(y_k + \alpha h f(y_k))$$

used to obtain approximate solutions to the differential equation

$$\frac{dy}{dt} = f(y) \text{ with } y(0) = y_0$$

(a) [7pts.] Derive an expansion for the leading term of the local truncation error.

(b) [3pts.] For what values of α is the numerical method globally second order?