Math 151A - Spring 2020

Final Exam : Instructions

- Name: ____
- Student ID: ____

- This exam is open notes/book, however, you may not use the internet. The only exception is for specific searches about the programming language, e.g., you may search for a command about plotting, printing, debugging, etc.

- Write legibly. No points will be given if we cannot understand your work. Submit all your codes and **show all** work needed to obtain your answers

- The exam is due tomorrow, Saturday, June 13, at 12 PM PDT.

Problem	**Points**
1	20
2	20
3	20
4	20
5	20

1. Let $f(x) = x^3 + 4x - 2$ and $I = [0, 1]$.

   (a) Prove that I contains a root of f

   (b) Show that finding the root x^* of f (i.e., $f(x^*) = 0$), is equivalent to finding the fixed-point of the function $g(x) = \dfrac{2 - x^3}{4}$. Is the fixed point unique?

   (c) State the fixed point iteration. Using any of the two error bounds for fixed point iterations we saw in class, estimate the number of iterations required for the fixed point iteration to give an approximation accurate to within 10^{-5} (you can consider any initial guess).

2. (Programming) Consider the following linearly convergent sequence:

$$p_0 = 0.5, \quad p_n = 3^{-p_{n-1}}, \quad n \geq 1$$

(a) Generate the first 100 terms of the sequences p_n, $\{\hat{p}_n\}$ using Aitken's Δ^2 method, and $p_2^{(n)}$ using Steffensen's method. Plot the values of $|p_{n+1} - p_n|$, $|\hat{p}_{n+1} - \hat{p}_n|$ for Aitken's, and $|p_2^{(n+1)} - p_2^{(n)}|$ for Steffensen's. Explain your findings.

3. Consider the function $f(x) = x^4 + \sin(\frac{\pi}{2}x)$ on the interval $[-1, 1]$.

   (a) Build the power series form of the polynomial interpolating f at the three nodes $x_0 = -1, x_1 = 0, x_2 = 1$.

   (b) Derive the Lagrange form of the polynomial interpolating f at the three nodes $x_0 = -1, x_1 = 0, x_2 = 1$.

(c) State the formula for the error when interpolating $f(x)$ for $x \in [-1, 1]$, and compute an upper bound for the error.

4. Derive the error for linear splines when using equidistant nodes $x_0, x_1, \ldots, x_n$.

5. Consider the integral $\int_{-2}^{2} x^3 e^x dx$. (This problem can optionally be done with programming)

   (a) For $n = 4$, approximate the integral using the Composite Trapezoidal rule

   (b) For $n = 4$, approximate the integral using the Composite Simpson's rule

   (c) For Composite Trapezoidal and Composite Simpson's, determine the value of n required to approximate the integral to within 10^{-4}.