UCLA Math 151A Winter 2022 − Midterm 2

Instructor: Victoria Kala, victoriakala@ucla.edu

Due March 5, 2022 by 9:00 am

This midterm covers topics covered in Lectures 10−19 and Sections 2.5−2.6, 3.1−3.6 of the textbook, namely accelerating convergence, Aitken's Delta-Squared Method, Steffensen's Method, zeros of polynomials, interpolation, Lagrange Interpolating Polynomials, data approximation with Neville's Method, divided differences, Newton's Form, Hermite Interpolation, Cubic Spline Interpolation, parametric cubic Hermite approximations, and Bézier Curves.

You may use your notes, textbook, homework, homework solutions, a calculator, and MATLAB codes provided to you in this course.

DO NOT discuss this exam with anyone other than the instructor. DO NOT use outside help such as your fellow classmates, tutors, other professors, the internet, etc. Students who are caught discussing this exam with others or are caught cheating will be reported to the Dean of Students.

Your solutions must be handwritten (e.g. pen and paper, digital handwritten on an iPad), no typed solutions will be accepted.

Each problem is worth 6 points (even if it has multiple parts) and will be graded according to the 6-point scale as described in the syllabus.

Solve the following exercises and submit your solutions through Gradescope by March 5, 2022 9:00am. No extensions will be given, do not email your midterm solutions to me or the TA. Solutions will be posted after the due date.

1. Prove the following Theorem:

   Suppose $x_0, x_1, ..., x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then for each x in $[a, b]$, a number $\xi(x)$ (generally unknown) between $\min\{x_0, x_1, ..., x_n\}$, and $\max\{x_0, x_1, ..., x_n\}$ and hence in (a, b), exists with

   $$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

   where $P(x)$ is the nth Lagrange interpolating polynomial given by

   $$P(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x), \quad L_{n,k}(x) = \prod_{i=0, i\neq k}^{n} \frac{(x - x_i)}{(x_k - x_i)}.$$

   Cite any important theorems used in your proof.

2. (a) Let $f(x) = \sin(\pi x)$ and $x_0 = 1, x_1 = 1.25$, and $x = 1.6$. Construct the Lagrange interpolating polynomial of degree at most two that passes through the points $(x_i, f(x_i))$, $i = 0, ..., 2$.

(b) Use your approximation in part (a) to approximate $f(1.4)$ and find the absolute error.

(c) Given $\max_{x\in[1,1.6]}|f^{(3)}(x)| \leq \pi^3$ and

$$\max_{x\in[1,1.6]}|(x-1)(x-1.25)(x-1.6)| \leq 0.01353,$$

use Theorem 3.3 to find an error bound for the approximation in part (a).

3. (a) Use Divided Differences to construct the Newton's form of the polynomial that passes through the following data: $f(-2) = 1,\ f(-1) = 0,\ f(1) = 5$.

(b) Use Divided Differences to construct the Hermite polynomial that passes through the following data: $f(0) = 0, f'(0) = 0, f(1) = 2, f'(1) = 3$.

4. (a) Consider the data $f(0) = 0, f(1) = -1, f(2) = -2$. Use the definition of a natural cubic spline to write the 8 equations that determine the coefficients a_j, b_j, c_j, d_j for $j = 0, 1$ of the spline. DO NOT SOLVE THE EQUATIONS.

(b) The solution to the system written in part (a) is $a_0 = c_0 = d_0 = c_1 = d_1 = 0$ and $b_0 = a_1 = b_1 = -1$. Write the equation of the natural cubic spline with these coefficients. Simplify as much as possible.

(c) Suppose the data $\{(x_i, f(x_i)\}_{i=1}^{n}$ lie on a straight line. What can be said about the natural and clamped cubic splines for the function f? (Hint: take a cue from parts (a) and (b) and Homework 5 Exercise 4)

5. State whether the following statements are **True** or **False**. **You do not need to show any work to receive credit**, but you may want to show work to arrive at the correct conclusion for some of these statements.

   (a) True or False? The Lagrange polynomial that interpolates the data $(-1, -1)$, $(0, 0)$, $(1, 1)$ will have degree 2.

   (b) True or False? Let $P(x)$ and $Q(x)$ be polynomials of degree at most n. If $x_1, x_2, ..., x_k$, with $k > n$, are distinct numbers with $P(x_i) = Q(x_i)$ for $i = 1, ..., k$, then $P(x) = Q(x)$ for all values of x.

   (c) True or False? A natural cubic spline satisfies the boundary conditions $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$.

   (d) True or False? The Lagrange form and the Newton form of the polynomial interpolating $(x_0, f(x_0)), ..., (x_n, f(x_n))$ are the same polynomial.

   (e) True or False? A cubic spline defined on an interval that is divided into n subintervals will require determining $4n$ constants.

   (f) True or False? Let $(0, 0)$ and $(1, 0)$ be the endpoints of a curve. If the corresponding guidepoints are $(1, 1)$ and $(0, 1)$, the Bézier Curve is then given by $x(t) = t$ and $y(t) = -t^2 + t$.