UCLA Math 151A Winter 2022 − Midterm 1

Instructor: Victoria Kala, victoriakala@ucla.edu

Due February 5, 2022 by 9:00 am

This midterm covers topics covered in Lectures 1−9 and Sections $1.1-1.4, 2.1-2.4$ of the textbook, namely calculus review, errors, convergence, the Bisection Method, Fixed-Point Iteration, Newton's Method, Order of Convergence, and Modified Newton's Method.

You may use your notes, textbook, homework, homework solutions, a calculator, and MATLAB codes provided to you in this course.

DO NOT discuss this exam with anyone other than the instructor. DO NOT use outside help such as your fellow classmates, tutors, other professors, the internet, etc. Students who are caught discussing this exam with others or are caught cheating will be reported to the Dean of Students.

Your solutions must be handwritten (e.g. pen and paper, digital handwritten on an iPad), no typed solutions will be accepted.

Each problem is worth 6 points (even if it has multiple parts) and will be graded according to the 6-point scale as described in the syllabus.

Solve the following exercises and submit your solutions through Gradescope by Feburary 5, 2022 9:00am. No extensions will be given, do not email your midterm solutions to me or the TA. Solutions will be posted after the due date.

1. Prove the following Theorem:

   Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$ for all $x \in [a,b]$. Suppose in addition that g' exists on (a,b) and that a constant $0 < k < 1$ exists with $|g'(x)| \leq k$ for all $x \in (a,b)$. Then for any number $p_0 \in [a,b]$, the sequence defined by

   $$p_n = g(p_{n-1}), \quad n \geq 1$$

   converges to the unique fixed point p in $[a,b]$.

   Cite any important theorems used in your proof.

2. (a) Briefly explain the Bisection Method algorithm. (Just a couple lines)

(b) Show that the function $f(x) = x^2 - 4x + 4 - \ln x = 0$ has a solution on the interval $[2, 4]$. Cite any important theorems used.

(c) Use the Bisection Method to find p_3 (i.e. 3 iterations) for $f(x) = x^2 - 4x + 4 - \ln x = 0$ on the interval $[2, 4]$. **You must show your handwritten work to receive full credit.** (You may check your solution with a calculator or MATLAB.)

3. (a) Suppose $f \in C[a, b]$. What assumptions are needed about $f(a), f(b)$ in order to use Bisection Method?

(b) The function $f(x) = \cos(x)$ has two zeros on the interval $[0, 2\pi]$. From part (a), can we use the Bisection Method for this function and this interval? If not, what can be modified so that we can use the Bisection Method? (**You do not need to solve or approximate the two zeros**, just explain to me how you would implement the Bisection Method for this particular function and interval.)

4. (a) State Newton's Method for solving $f(x) = 0$ with an initial guess p_0.

(b) Use Newton's method to determine p_3 (i.e. 3 iterations) for $f(x) = x^3 + 3x^2 - 1 = 0$ and $p_0 = 1$. **You must show your handwritten work to receive full credit.** (You may check your solution with a calculator or MATLAB).

5. Consider the function $f(x) = (x+3)(x-1)^2$.

   (a) Identify the zeros of $f(x)$ and their multiplicities.

   (b) If we were to use Newton's Method to approximate each of these zeros, should we expect quadratic convergence? Briefly explain. If not, what method should we use instead to guarantee quadratic convergence? (**You do not need to perform any calculations for this part**, just explain your reasoning.)

6. State whether the following statements are **True** or **False**. You do not need to show any work to receive credit, but you may want to show work to arrive at the correct conclusion for some of these statements.

   (a) True or False? Let p^* be an approximation to p. The absolute error is $|p - p^*|$ and the relative error is $\dfrac{|p - p^*|}{|p|}$, provided $p \neq 0$.

   (b) True or False? The Bisection Method always converges to a solution for $f(x) = 0$ on the interval $[a, b]$ if f is continuous and $f(a), f(b)$ have opposite signs.

   (c) True or False? The error bound for Bisection Method implies the method converges to a solution within 10^{-3} in at most 8 iterations for the function $f(x) = e^x - x^2 + 3x - 2 = 0$ on the interval $[0, 1]$.

   (d) True or False? The Fixed-Point Iteration for $g(x) = 2^{-x}$ converges to the unique fixed point of $g(x)$ for any $p_0 \in \left[\dfrac{1}{3}, 1\right]$.

   (e) True or False? Newton's Method converges for any initial guess p_0.

   (f) True or False? If Newton's method converges, it converges quadratically.