$9x - x^2 - 3xy$ $= \begin{bmatrix} 9-2x-3y & -3x \\ 2y & -6+2x \end{bmatrix}$

$-6y + 2xy$

(0, 0)

(9, 0)

(3, 2)

1. Let x and y represent two animal populations which satisfy

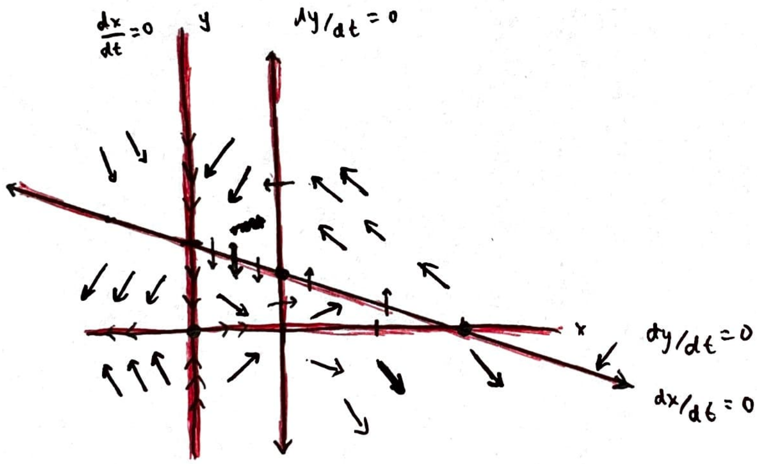
$$\frac{dx}{dt} = x(9 - x - 3y)$$
$$\frac{dy}{dt} = y(-6 + 2x)$$

(a) (5 points) What is the relationship between x and y? How does x grow in the absence of y? How does y grow in the absence of x?

x is the prey, while y is a predator. Without y, x will grow akin to logistic growth, so depending on the starting population, x will decrease or increase to the carrying capacity of 9. However, y will experience exponential decay in the abscence of x. Without x, y will also die off.

(b) (5 points) Sketch the nullclines and direction arrows of the system.

Consider $x=1, y=1$
$\Rightarrow dx/dt = 5, \; dy/dt = -4$



(c) (4 points) Find the eigenvalues of the interior critical point.

$dx/dt = 9x - x^2 - 3xy$; $dy/dt = -6y + 2xy$.

$(x_c, y_c) \Rightarrow 9 - x - 3y = 0, \; 2x = 6 \Rightarrow x = 3, \; y = 2 \Rightarrow (x_c, y_c) = (3, 2)$

$J(x,y) = \begin{bmatrix} 9-2x-3y & -3x \\ 2y & -6+2x \end{bmatrix} \Rightarrow J(3,2) = \begin{bmatrix} -3 & -9 \\ 4 & 0 \end{bmatrix}$. $\det(\lambda I - J) = 0$

$\Rightarrow \begin{vmatrix} \lambda+3 & 9 \\ -4 & \lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 + 3\lambda + 36$

~~$\lambda^2 + 3\lambda + 27 = 0$~~. By Quadratic formula, $\lambda = \dfrac{-3 \pm \sqrt{9 - 4(36)}}{2}$

$\cancel{= -\frac{3}{2} \pm i \frac{\sqrt{87}}{2}}$ $\Rightarrow \boxed{-\frac{3}{2} \pm i \frac{\sqrt{135}}{2}}$

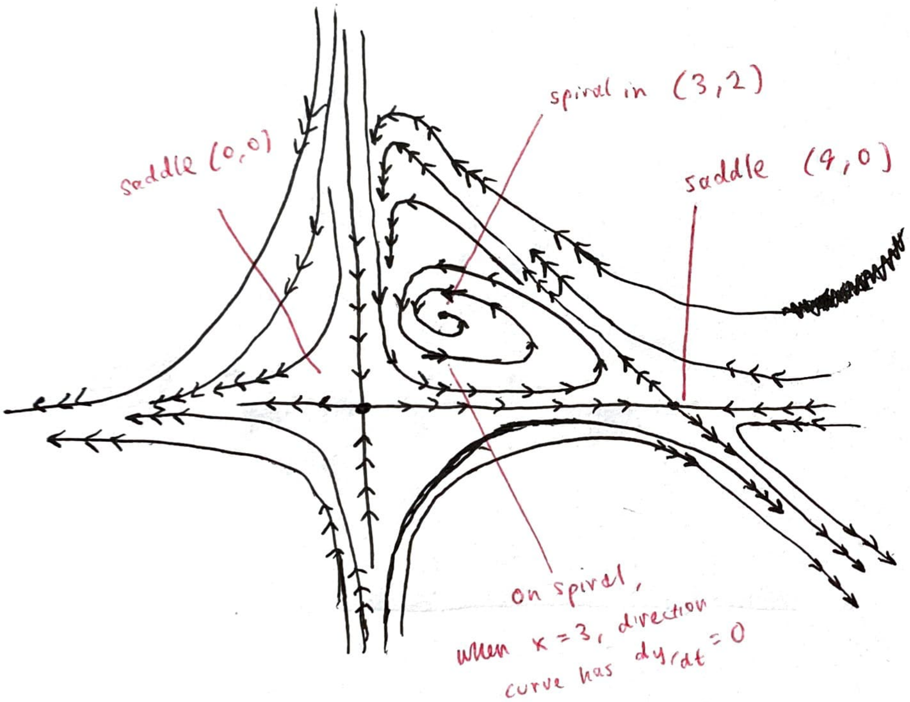
(d) (7 points) Sketch the general solution. Be detailed.

We find eigenvalues of saddle pts to examine detailed behavior around saddles.

For $(0,0)$, $J = \begin{bmatrix} 9 & 0 \\ 0 & -6 \end{bmatrix}$; by inspection, it's diagonal w/ $\lambda = 9, -6$ and $\underbrace{\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\text{dominant}}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

For $(9,0)$, $J = \begin{bmatrix} -9 & -27 \\ 0 & 12 \end{bmatrix} \Rightarrow (\lambda+9)(\lambda-12) = 0 \Rightarrow \lambda = -9, 12$. For $\lambda = 12$, $\begin{bmatrix} -21 & -27 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \vec{v}_1 \end{bmatrix} = \vec{0}$

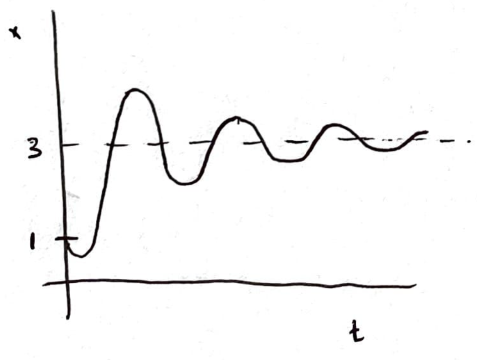
$\Rightarrow \vec{v}_1 = \begin{bmatrix} 27 \\ -21 \end{bmatrix} = \begin{bmatrix} 9 \\ -7 \end{bmatrix}$ as dominant eigenvector. For $\lambda = -9$, $\begin{bmatrix} 0 & -27 \\ 0 & 21 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

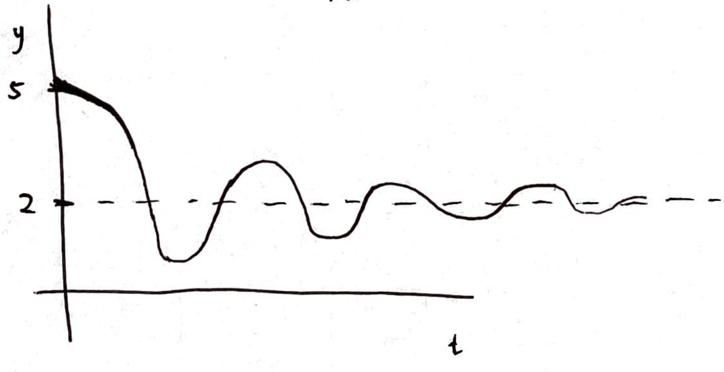


(e) (4 points) Sketch $x(t)$ and $y(t)$ if $x(0) = 1$, $y(0) = 5$.

$dx/dt = 1(9 - 1 - 15) = -7$

$dy/dt = 5(-6+2) = -20 < 0$





2. (8 points) Determine whether the polynomial has any roots with positive real parts.

$$\overset{a_0}{\lambda^4} + \overset{a_1}{4\lambda^3} + \overset{a_2}{5\lambda^2} + \overset{a_3}{6\lambda} + \overset{a_4}{2}.$$

We use Hurwitz Matrices! Recall

$$H_1 = [a_1],\quad H_2 = \begin{bmatrix} a_1 & a_3 \\ a_0 & a_2 \end{bmatrix},\quad H_3 = \begin{bmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{bmatrix},\quad H_4 = \begin{bmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{bmatrix}$$

and we seek to show that iff H_i all have positive discriminant, then our polynomial has all negative real part sol'ns.

We use $a_0 = 1$, $a_1 = 4$, $a_2 = 5$, $a_3 = 6$, $a_4 = 2$ and $a_5, a_6, a_7 = 0$.

✓ $\det(H_1) = 4 > 0$ trivially

✓ $\det \begin{vmatrix} 4 & 6 \\ 1 & 5 \end{vmatrix} = 20 - 6 = 14 > 0$

✓ $\det \begin{vmatrix} 4 & 6 & 0 \\ 1 & 5 & 2 \\ 0 & 4 & 6 \end{vmatrix} = 4\begin{vmatrix} 5 & 2 \\ 4 & 6 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ 0 & 6 \end{vmatrix} = 4(30-8) - 6(6-0) = 120 - 32 - 36 = 52 > 0.$

✓ $\det \begin{vmatrix} 4 & 6 & 0 & 0 \\ 1 & 5 & 2 & 0 \\ 0 & 4 & 6 & 0 \\ 0 & 1 & 5 & 2 \end{vmatrix} = \cancel{\begin{vmatrix} 5 & 5 & 2 \\ 4 & 4 & 6 \\ 7 & 1 & 5 \end{vmatrix}} \cancel{6\begin{vmatrix} 1 & 2 & 6 \\ 0 & 6 & 0 \\ 0 & 5 & 2 \end{vmatrix}} = 4\begin{vmatrix} 5 & 2 & 0 \\ 4 & 6 & 0 \\ 1 & 5 & 2 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 & 0 \\ 0 & 6 & 0 \\ 0 & 5 & 2 \end{vmatrix}$

$\cancel{4\left(\begin{vmatrix} 4 & 6 \\ 1 & 5 \end{vmatrix} - 5\begin{vmatrix} 0 & 6 \\ 0 & 5 \end{vmatrix} + 2\begin{vmatrix} 0 & 6 \\ 0 & 5 \end{vmatrix}\right) - 6\left(\begin{vmatrix} 6 & 0 \\ 5 & 2 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix}\right)}$

$\cancel{= 4(1+4)}$ $= 4\left(5\begin{vmatrix} 6 & 0 \\ 5 & 2 \end{vmatrix} - 2\begin{vmatrix} 4 & 0 \\ 1 & 2 \end{vmatrix}\right) - 6\left(1\begin{vmatrix} 6 & 0 \\ 5 & 2 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix}\right)$

$= 4(60 - 16) - 6(12)$

$= 176 - 72 = \underline{104} > 0$

Thus det of 4 Hurwitz matrices are all positive, so this polynomial has <u>no roots</u> with positive real parts.

3. Suppose x, y, z satisfy the competing species equations

$$\begin{aligned}\frac{dx}{dt} &= x(6-2x-3y-z) = f \\ \frac{dy}{dt} &= y(7-2x-3y-2z) = g \Rightarrow \\ \frac{dz}{dt} &= z(5-2x-y-2z) = h\end{aligned}$$

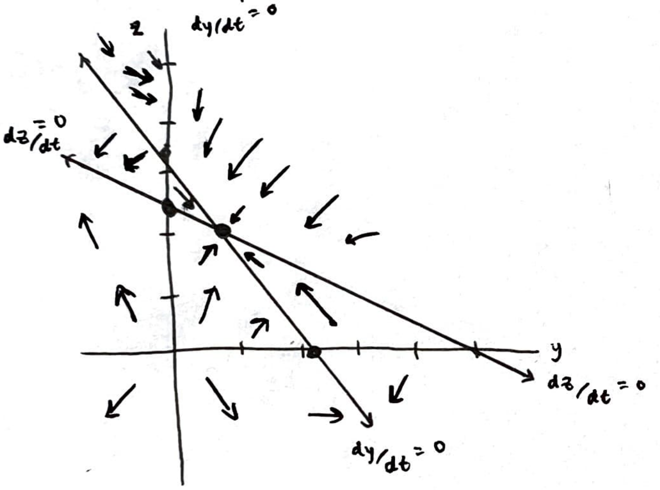
$7 - 3y - 2z$

$5 - y - 2z$

(a) (6 points) Find the critical point $(0, y_c, z_c)$ where $y_c, z_c > 0$, and sketch the nullclines and direction arrows in the yz-plane.

Suppose $x = 0$. $\Rightarrow$ $3y + 2z = 7$; ~~2y~~ $y + 2z = 5$ $\Rightarrow$ $2y = 2$, $y = 1, z = 2$

critical pt = $(0, 1, 2)$



(b) (6 points) Determine if $(0, y_c, z_c)$ is stable.

In the yz plane, $(0, y_c, z_c)$ does appear to be stable. We look ~~out~~ at the Jacobian of the system: $\frac{\partial f}{\partial x} = 6 - 4x - 3y - z$, $\frac{\partial f}{\partial y} = -3x$, $\frac{\partial f}{\partial z} = -x$, $\frac{\partial g}{\partial x} = -2y$, $\frac{\partial g}{\partial y} = 7 - 2x - 6y - 2z$; $\frac{\partial g}{\partial z} = -2y$

and similarly for $\partial h/\partial x = -2z$, $\partial h/\partial y = -z$, $\partial h/\partial z = 5 - 2x - y - 4z$

$$J(0, y_c, z_c) = \begin{bmatrix} 6-4x-3y-z & -3x & -x \\ -2y & 7-2x-6y-2z & -2y \\ -2z & -z & \cancel{5-2x-y-z}\ \cancel{5-y-4z}\ 5-2x-y-4z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -3 & -4 \\ -4 & -2 & -4 \end{bmatrix}$$

$\det(\lambda I - J) = (\lambda - 1)\left((\lambda+3)(\lambda+4) - 8\right) = 0$.

Clearly, $\lambda = 1$ is an eigenvalue, but $Re(\lambda = 1) = 1 > 0$, so this indicates that $(0, y_c, z_c)$ is not stable!

(c) (8 points) Determine if the critical point $(x_c, 0, 0)$ is stable, where $x_c > 0$.

Recall from part (b) we showed that

$$J(x,y,z) = \begin{bmatrix} 6-4x-3y-z & -3x & -x \\ -2y & 7-2x-6y-2z & -2y \\ -2z & -z & 5-2x-y-4z \end{bmatrix}$$

We find x_c where $(x_c, 0, 0)$ is a critical pt $\Rightarrow$ $6 - 2x_c = 0 \Rightarrow x_c = 3$.

$$J(x_c, 0, 0) = \begin{bmatrix} -6 & -9 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

This is triangular, so we know eigenvalues are on the diagonal. ~~Thus~~

~~thus ex~~

Since one of the eigenvalues is 1, we know that the critical point is unstable ~~and~~ because $1 > 0$.

Thus $(3,0,0)$ is unstable!

$rN(1-\frac{N^3}{k})$ =

4. Recall from Homework 3 that Sea Lions grow the generalized logistic equation $\frac{dN}{dt} = rN(1 - (\frac{N}{K})^p)$, where $p \approx 3$.

Suppose a population distribution of Sea Lions $u(x,t)$ follows

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - u(8-u^3).$$

(a) (4 points) Set $u(x,t) = U(x-kt)$ and $V = U'$. Find a 2×2 system of ODE's for U, V.

Since $\frac{\partial u}{\partial t} = -kU'$ and $\frac{\partial u}{\partial x} = u'$, we have

$-kU' = \frac{1}{2}u'' - u(8-u^3)$.

Substituting $V = U'$

$\Rightarrow \quad -kV = \frac{1}{2}V' - U(8-u^3) \Rightarrow 2U(8-u^3) - 2kV$

$\Rightarrow$ Our system is $\begin{cases} V' = 2U(8-u^3) - 2kV \\ U' = V \end{cases}$

(b) (5 points) Find the critical points, and the Jacobian at each critical point.

Our critical points are $V=0$ and $\Rightarrow 2U(8-U^3)=0$, so we take $U=0$, $U=2$ for critical pts of $(0,0)$ and $(2,0)$. $\frac{\partial V'}{\partial U} = 2(8-U^3) + 2U(-3U^2)$

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ 16-8U^3 & -2k \end{bmatrix}. \Rightarrow J(0,0) = \begin{bmatrix} 0 & 1 \\ 16 & -2k \end{bmatrix}, \quad J(2,0) = \begin{bmatrix} 0 & 1 \\ -48 & -2k \end{bmatrix}$$

(c) (5 points) Find the minimum value of k for which a traveling wave solution exists, which could correspond to the population distribution of seal lions.

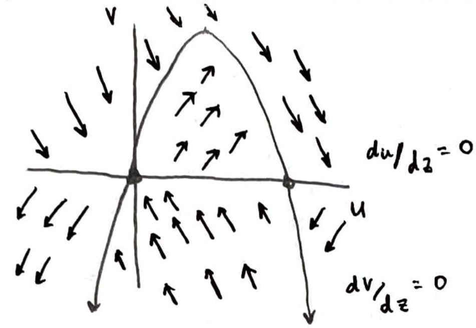
Consider eigenvalues of $J(0,0)$ and $J(2,0)$, using ~~$\lambda^2 - T\lambda +$~~ $\lambda^2 - T\lambda + D$

$J(0,0) \Rightarrow \lambda^2 + 2k\lambda - 16 \Rightarrow \frac{-2k \pm \sqrt{4k^2+64}}{2}$, always a saddle bc $\sqrt{4k^2+64} > |-2k|$

$J(2,0) \Rightarrow \lambda^2 + 2k\lambda + 48 = 0 \Rightarrow \lambda = \frac{-2k \pm \sqrt{4k^2 - 4(48)}}{2}$. Note that this is always stable, but may not be real. So, we take $4k^2 - 4(48) \geq 0 \Rightarrow k^2 \geq 48 \Rightarrow$

$k = \sqrt{48} = \underline{4\sqrt{3}}$ as our minimum wave speed, when we seek real sol'ns

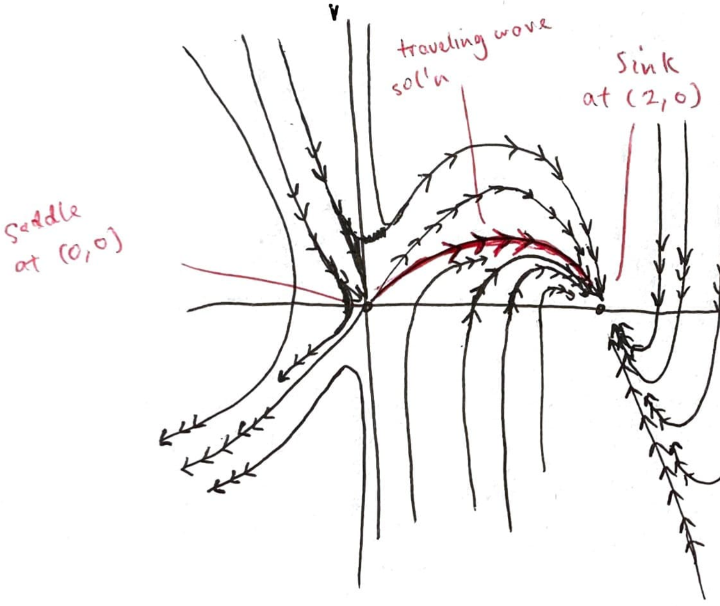
(d) (5 points) Sketch the nullclines and direction arrows of the system.



Consider $V > 0$, $u < 0$

$\Rightarrow \frac{du}{dt} > 0$,

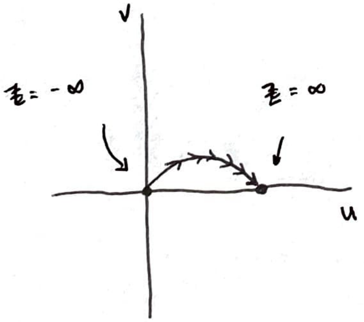
(e) (7 points) Sketch the general solution to your system above in the UV-plane, where both critical points have real eigenvalues. Don't draw vague or incomplete solutions.



(2,0) is a sink, but all curves should approach tangent to some eigenvector.

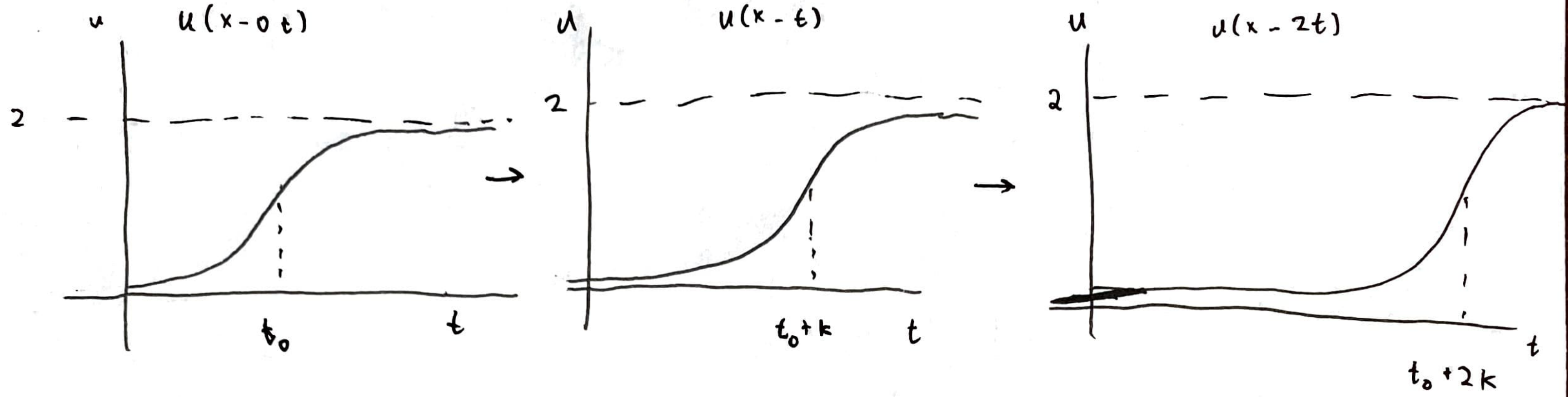
(f) (3 points) Indicate the **specific** solution in your picture above, which corresponds to a traveling wave solution.

The sol'n colored in red is our specific traveling wave solution.



where $z = x - kt$

(g) (5 points) Graph the solution $U(x - kt)$ for $t = 0, 1, 2$ for the traveling wave.



(h) (3 points) Why did the eigenvalues at the critical points have to be real?

The eigenvalues at (0,0) were always real. However, for our critical pt (2,0) we sought ~~real~~ critical pts w/ real eigenvalue to plot the case of a nodal sink. Though sufficiently small spirals in the $+U$ and $\pm V$ plane are technically realistic, a large enough spiral may cause parts of the spiral to reach into $-U$ side on the plane. By taking ~~real e~~ critical pts w/ real eigenvalues, we ~~ensure~~ guarantee that the solution is physically realistic.

5. (10 points) The lexical similarities between the Numic languages are boxed in the table on the next page. The abbreviations correspond to the following map. Use both the map and the lexical similarities to explain **when** the Numic spread must have happened. Be specific in how you reach this conclusion... you have a whole page.

→ Lexical similarities are relevant to population spread because as people move, languages also change. Swadesh's list of 100 words given an approx. to how similar two languages are, and given that two languages share some common parent language, we can find out how many years ago they began to diverge w/ the formula $t = 500 \frac{\ln c}{\ln 0.85}$, where C is the % shared in the 100 words

→ Given this information, we can use current linguistic data to analyze the movement of Numic people ~~that used to live~~ in Southern California. On the map, we see that the pre-spread Numic peoples can be divided into 3: western, northern, and eastern Numic, speaking Mono (Mn), Tumpisha Shoshini (TSh), and Kawaiisu (Kw) respectively. These three languages have relatively low similarity, pairwise 59, 54, 52, which indicates they began to diverge close to $t \approx 500 \frac{\ln 0.54}{\ln 0.85} = 1900$ years ago. But more interestingly, if we look at differences within ~~subgroups~~, geographic subgroups, we see that though Northern Paiute (NP) is far more spread out than the 3 pre-spread Numic groups, NP shares high lexical similarity w/ its western ~~northern~~ neighbor Mn of about 77, which indicates spread in Northern Numic began $t \approx 500 \frac{\ln 0.77}{\ln 0.85}$ or 800 years ago. If we do similar calculations w/ ~~Center~~ Northern Numic languages, comparing TSh with Shoshoni (Sh), we see divergence of $t \approx 500 \frac{\ln 0.87}{\ln 0.88} = 430$ years ago. far more recent compared to 1900 years! Finally for eastern numic, we have Kw's similarity to Comanche and Southern Paiute ~~to~~ yielding results of $500 \frac{\ln 0.75}{\ln 0.85} = 885$ years and $500 \frac{\ln 0.79}{\ln 0.85} = 725$ years. Thus, combining western, northern, and eastern numic spread, we might guess that numic spread began 400 to 800 years ago.

→ In just those few years, compared to the 2,000 spent in only eastern California, this indicates how rapidly Numic populations were able to spread. This goes hand-in-hand w/ the commentary provided about Numic basketry and the competitive advantages gained from seed harvesting as Numic technology improved.