Math 134
Fall 2016
Midterm Exam 2
11/9/2016
Time Limit: 50 Minutes

This is a closed book test. Do all work on the sheets provided. There are four problems. Problem 1-3 consist of 100 points, Problem 4 is optional with 20 bonus points. Your total score will be at most 100 points.

93

75

1. (30 points) Consider the following nonuniform oscillator (**A flow on the circle**),

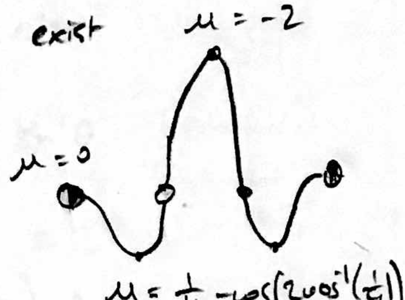
$$\dot{\theta} = \mu + \cos\theta + \cos 2\theta$$

(1) Draw the phase portrait as function of the control parameter μ (Flows on the circle).

(2) Classify the bifurcations that occur as μ varies.

(3) Find all the bifurcation values of μ.

(1) Consider that when $\frac{d\dot{\theta}}{d\theta} = 0$, there exist a bifurcation point (μ, θ).



$$\rightarrow \frac{d\dot{\theta}}{d\theta} = 0 = -\sin\theta - 2\sin 2\theta$$

$$= -\sin\theta - 4(\sin\theta\cos\theta)$$

$$= -\sin\theta(1 + 4\cos\theta) = 0$$

when $\theta = k\pi$ and $\theta = \pm\cos^{-1}(-\frac{1}{4}) \in (\frac{\pi}{2}, \pi)$ $(-\pi, -\frac{\pi}{2})$

Restrict to $[-\pi, \pi]$.

Then: determine μ s.t. $\dot{\theta} = 0$.

$$\mu + \cos(\theta) + \cos(2\theta) = 0$$

$$\mu = -\cos(\theta) - \cos(2\theta)$$

$\dot{\theta}(\mu, 0) = 0 = \mu + 1 + 1 \rightarrow \mu = -2$

$\dot{\theta}(\mu, \pi) = 0 = \mu - 1 + 1 \rightarrow \mu = 0$

$\dot{\theta}(\mu, \cos^{-1}(-\frac{1}{4})) = 0 = \mu + (-\frac{1}{4}) + \cos(2\cos^{-1}(-\frac{1}{4}))$

$\mu = \frac{1}{4} - \cos(2\cos^{-1}(\frac{1}{4}$

$\dot{\theta}(\mu, -\cos^{-1}(-\frac{1}{4})) = 0 = \mu + (-\frac{1}{4}) + \cos(2\cos^{-1}(-\frac{1}{4}))$

(3) $\mu = -2, 0, \frac{1}{4} - \cos(2\cos^{-1}(-\frac{1}{4}))$ ■

~~Phase Portrait~~

$\mu = -2 \rightarrow$ Saddle-Node Bifurcation

(2) $\mu = 0 \rightarrow$ Saddle-Node Bifurcation

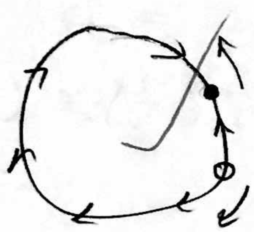
-2

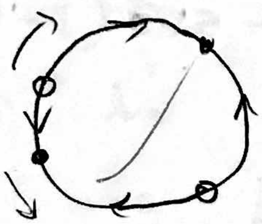
b/c Sinusoidal Function

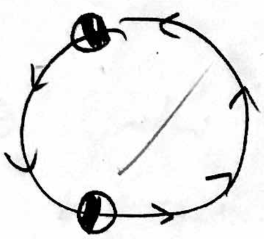
$\mu = \frac{1}{4} - \cos\left(2\cos^{-1}\left(-\frac{1}{4}\right)\right) = \gamma = ?$ $\boxtimes$

$\rightarrow$ Saddle-Node Bifurcation

(1) $\mu \in (-2, 0)$ $\quad$ $\mu \in (0, \gamma)$ $\quad$ $\mu = \gamma$





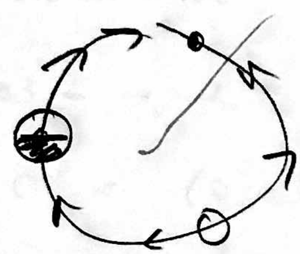


$\mu > \gamma$

$\mu < -2$

$\mu = -2$

$\mu = 0$



$\boxtimes$

-3

 33

2. (40 points) Consider the following 2-D nonlinear system.

$$\begin{cases} \dot{x} = x - y \\ \dot{y} = x^2 - 4 \end{cases}$$

(1) Find all the fixed points.

(2) At each fixed point, show the linearized system accordingly. eigenvectors? -4

(3) Classify all the fixed points.

(4) Sketch the neighboring trajectories.

x^2

(1) $\dot{x} = 0 = x - y \rightarrow y = x$

$\dot{y} = 0 = x^2 - 4 \rightarrow x = \pm 2$

The fixed points $\tilde{X}$ are located at $(2,2)$, and $(-2,-2)$

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2x & 0 \end{pmatrix}$$

(3) $\rightarrow A(x = y = -2) = \begin{pmatrix} 1 & -1 \\ -4 & 0 \end{pmatrix}$ $\tau = 1$, $\Delta = -4$

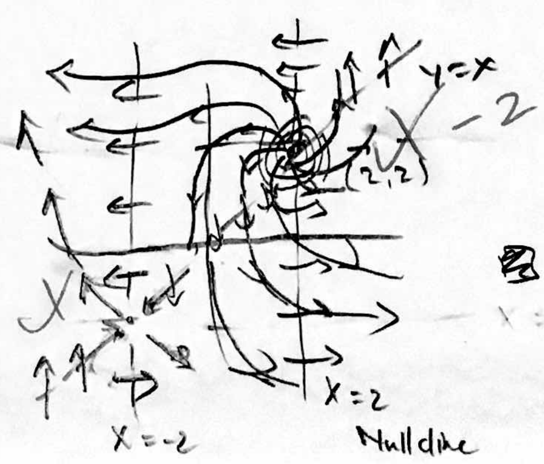
$\tilde{X} = (-2,-2)$: saddle point

$\rightarrow A(x = y = 2) = \begin{pmatrix} 1 & -1 \\ 4 & 0 \end{pmatrix}$ $\tau = 1$, $\Delta = 4$

$\tau < 2\sqrt{\Delta} \rightarrow$ Unstable Spiral.

(2)

(4)



$(1 - \lambda)(-\lambda) - 4 = 0$

$\rightarrow p(\lambda) = \lambda^2 - \lambda - 4 = 0$

$$\lambda = \frac{1 \pm \sqrt{1 - 4(-4)}}{2} = \frac{1 \pm \sqrt{17}}{2}$$

$\lambda_1 = \frac{1 + \sqrt{17}}{2} > 0$ $\lambda_2 = \frac{1 - \sqrt{17}}{2} < 0$

3. (30 points) Consider the following 2-D linear system with parameter ϵ.

$$\begin{cases} \dot{x} = \epsilon x + y \\ \dot{y} = -x - 2y \end{cases}$$

(1) What is the first **nonzero** digit in your ID?

(2) If ϵ equals the first **nonzero** digit in your ID, classify the fixed point of (0,0).

(3) Draw the phase portrait.

30

(1) 7 [redacted]

(2) $\dot{x} = 7x + y = 0 \rightarrow y = -7x$

$\dot{y} = -x - 2y = 0 \rightarrow y = -\frac{1}{2}x$

$A = \begin{pmatrix} 7 & 1 \\ -1 & -2 \end{pmatrix} \rightarrow \tau = 5$

$\Delta = -13$

(0,0) → Saddle-Node [redacted]

(3)

$p(\lambda) = \lambda^2 - 5\lambda - 13 = 0$

$\rightarrow \lambda = \frac{5 \pm \sqrt{25-4(-13)}}{2}$

$\lambda_i = \frac{5 \pm \sqrt{77}}{2}$

$\begin{array}{r} 13 \\ \times 4 \\ \hline 52 \\ +25 \\ \hline 77 \end{array}$

$\begin{pmatrix} 7-\lambda & 1 \\ -1 & -2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\lambda_1 \approx \frac{14}{2} = 7$

$\lambda_2 \approx -2$ (Approx.)

$\rightarrow (7-\lambda)x + y = 0$

$-x - (2+\lambda)y = 0$

$\rightarrow x = -(2+\lambda)y$

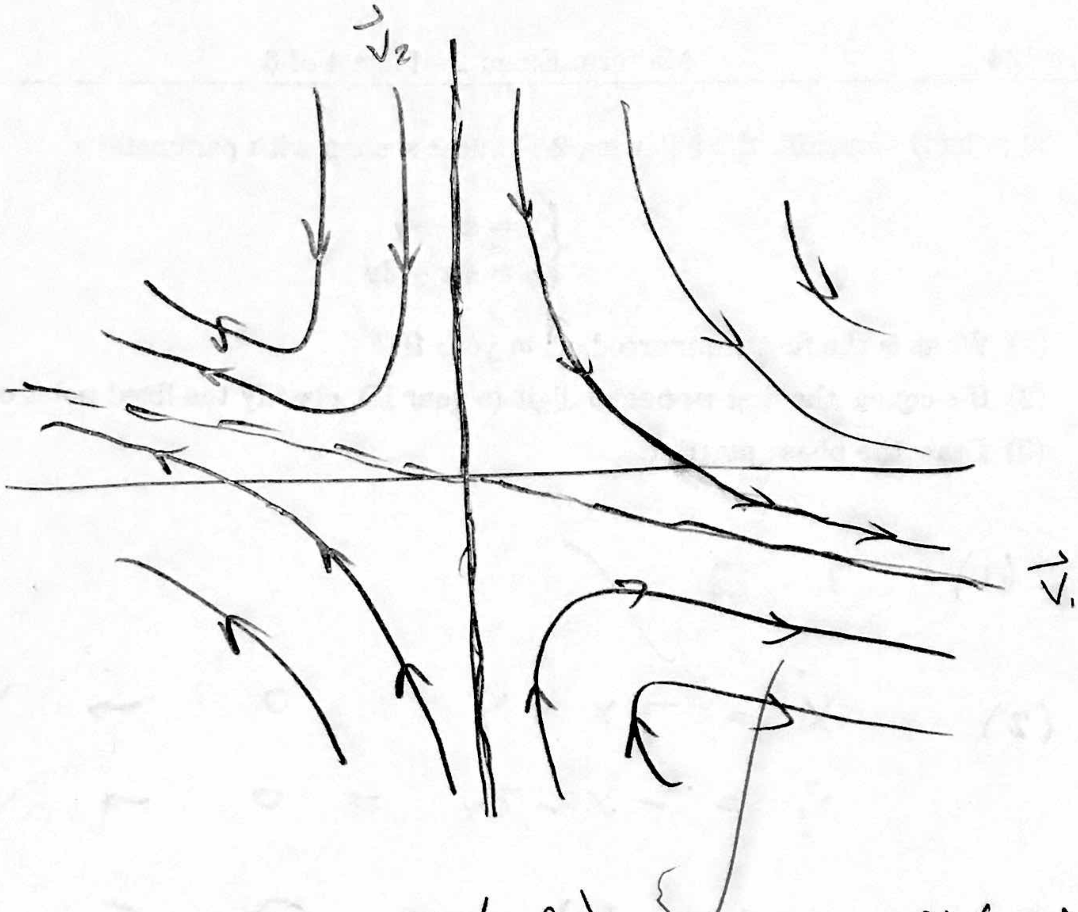
$\rightarrow V_1 = \begin{pmatrix} -(2+\lambda) \\ 1 \end{pmatrix}$

$\lambda_1 = 7$

$V_1 \approx \begin{pmatrix} -9 \\ 1 \end{pmatrix}$

$V_2 \approx \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\lambda_2 = -2$

Phase Portrait (Approximate)



$$\underline{X}(t) \approx C_1 e^{7t}\begin{pmatrix}-9\\1\end{pmatrix} + C_2 e^{-2t}\begin{pmatrix}0\\1\end{pmatrix} \quad \boxtimes$$

BONUS

4. (20 points) Revisit Problem 3.

$$\begin{cases} \dot{x} = \epsilon x + y \\ \dot{y} = -x - 2y \end{cases}$$

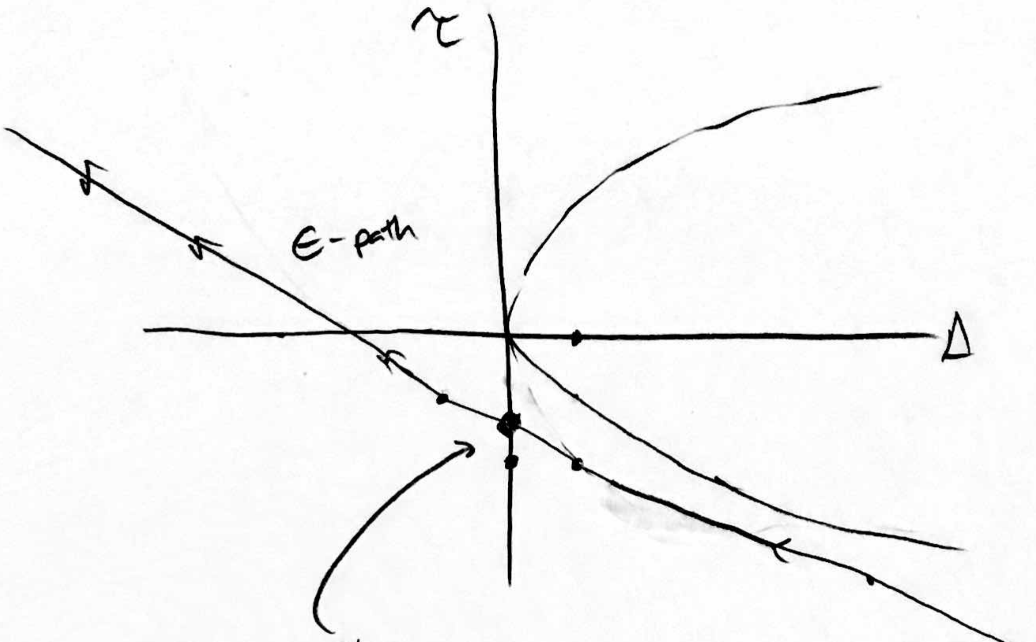
Discuss all possible values of ϵ and classify the fixed point at $(0,0)$ accordingly.

$$\dot{x} = \epsilon x + y = 0 \rightarrow y = -\epsilon x$$

$$\dot{y} = -x - 2y = 0 \rightarrow y = -\tfrac{1}{2}x$$

$$A = \begin{pmatrix} \epsilon & 1 \\ -1 & -2 \end{pmatrix} \rightarrow \begin{matrix} \tau = \epsilon - 2 \\ \Delta = 1 - 2\epsilon. \end{matrix}$$

Consider the trajectory of ϵ in the (τ, Δ) plane.



As $\epsilon \to \infty$ from $-\infty$, the trajectory changes the classification of $\tilde{X} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ from stable node to saddle node. ∎

When $\Delta = 0$

$\Rightarrow \epsilon = \frac{1}{2}$ and $\tilde{X}$: stable line. v_1, v_2.

(non-isolated)

$$A = \begin{pmatrix} 1/2 & 1 \\ -1 & -2 \end{pmatrix} \rightarrow p(\lambda) = \lambda^2 + 3/2\,\lambda \rightarrow \lambda = 0, -3/2 \;//$$

Grade Table (for teacher use only)

Question	Points	Score
1	0	25
2	40	33
3	30	30
4	20	5
Total:	90	93