Math 134
Fall 2016
Midterm Exam 1
10/17/2016
Time Limit: 50 Minutes

This is a closed book test. Do all work on the sheets provided.

Grade Table (for teacher use only)

Question	Points	Score
1	40	40
2	20	10
3	20	8
4	20	14
Total:	100	72

1. (40 points) Consider the following nonlinear differential equation,

$$\dot{x} = e^x \cos(x), \quad x(0) = x_0$$

(1) State the **Existence and Uniqueness Theorem** for general initial value problem $\dot{x} = f(x), x(0) = x_0$.

(2) Show that the above initial value problem has a unique solution for any x_0.

(3) **Sketch** the phase portrait, find all fixed points for the above system and classify their stability.

(4) Use linear stability analysis to show the stability of point $\frac{\pi}{2}$.

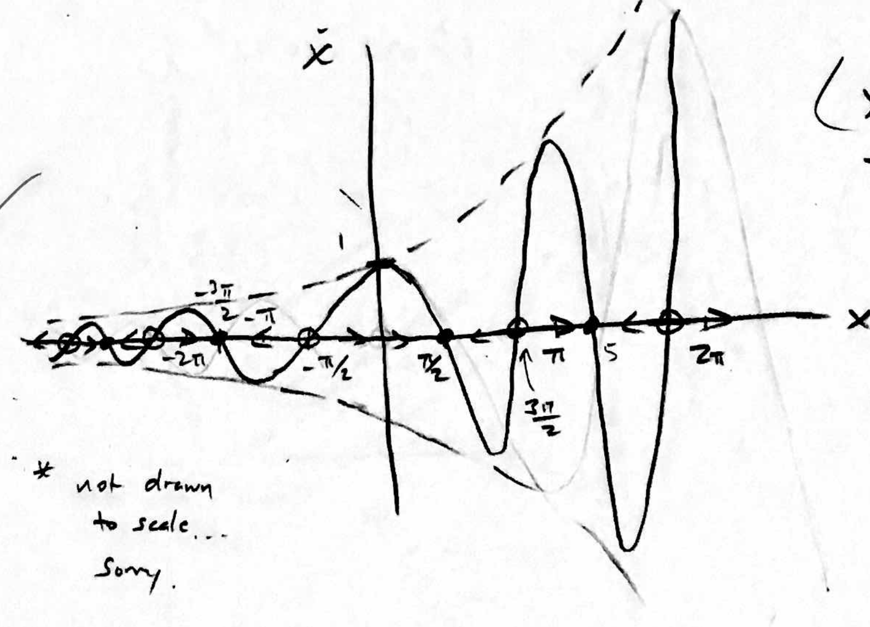
(1) Suppose $\dot{x} = f(x)$ is a first-order system with initial condition $x(0) = x_0$. Then if $f \in C^1$, then $\exists!$ solution to $\dot{x} = f(x)$ in an open interval I (around the origin). ∎

(2) Consider that $e^x \cos(x)$ is continuous, because e^x : cts. and $\cos(x)$: cts. Then,

$f'(x) = \cos(x) e^x - e^x \sin x = e^x(\cos x - \sin x)$

is continuous $\forall x \in \mathbb{R}$ as well. By uniqueness and existance, $\exists!$ solution in some interval I to $\dot{x} = e^x \cos x$, for any initial condition $x(0) = x_0$. ∎

(3)



* not drawn to scale... Sorry.

$\left(x_k^* = (2k+1)\frac{\pi}{2} \quad k \in \mathbb{Z} \right)$

$x_k^* = (4k+1)\frac{\pi}{2}$: stable

$x_k^* = (4k-1)\frac{\pi}{2}$: unstable

∎

(4) $\frac{d\dot{x}}{dx} = f'(x) = \cos x \, e^x - \sin x \, e^x$

$$f'\left(\frac{\pi}{2}\right) = \cancel{\cos\frac{\pi}{2}}^{\,0} e^{\pi/2} - \sin\frac{\pi}{2} e^{\pi/2}$$

$$= -1 e^{\pi/2} < 0$$

$\Rightarrow \quad x^* = \frac{\pi}{2}$: stable equilibrium point. $\blacksquare$

2. (20 points) Consider the following nonlinear system with parameter r.

$$\dot{x} = r + x - \ln(1 + x)$$

(1) Find the critical value r_c and the fixed point x^* such that the bifurcation occurs.

(2) **Sketch** the phase portraits for $r < r_c$, $r = r_c$ and $r > r_c$.

(3) Classify the bifurcation.

(4) **Sketch** the bifurcation diagram.

(1) $\dot{x} = 0 = r + x^* - \ln(1+x^*)$

$\Rightarrow r + x^* = \ln(1+x^*)$

$r = x^* + \ln(1+x^*)$

Suppose the derivatives are tangent curves

$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3!} \cdots$

$\approx x - \frac{x^2}{2}$

(T)

$x^* = \tilde{x}$

$r \approx 2\tilde{x} - \frac{1}{2}\tilde{x}^2$

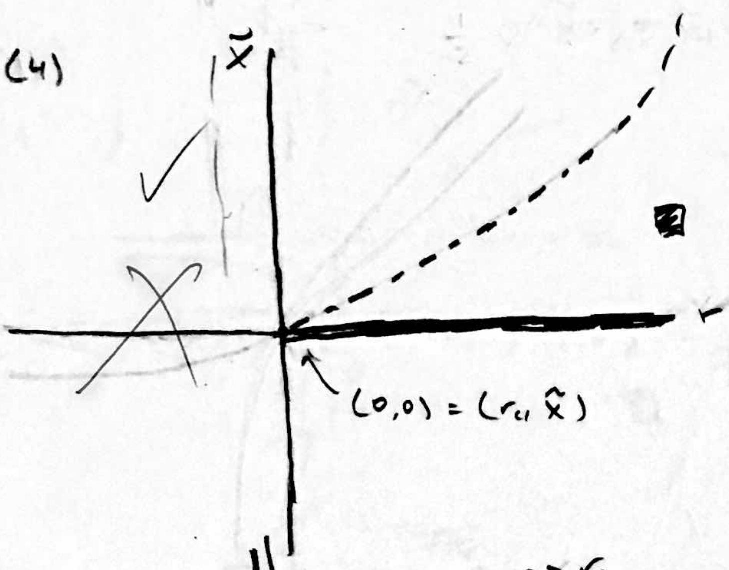
$\frac{d}{dx^*} r + x^* = \frac{d}{dx^*} \ln(1+x^*)$

$\rightarrow 1 = \frac{1}{1+x^*}$

$\rightarrow x^* = 0$

$\rightarrow r_c = 0$

(4)



(3) $f'(x) = 1 - \frac{1}{1+x^*}$

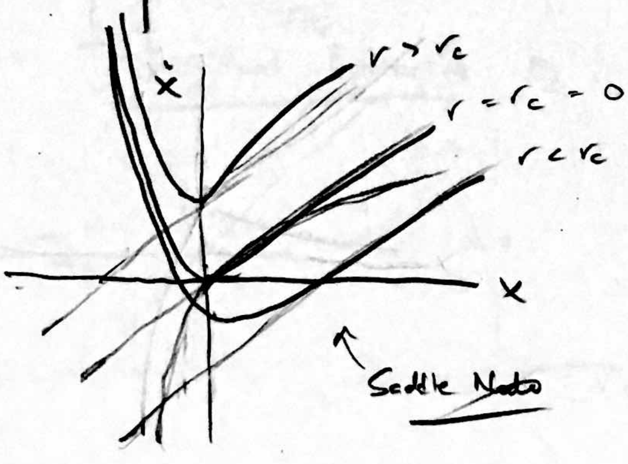
$f'(-\frac{1}{2}) = -1 < 0$

$f'(\frac{1}{2}) = \frac{1}{3} > 0$

$\rightarrow$ Saddle-Node Bifurcation

$\approx$ $\underline{r \pm x^2}$

(2)



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3. (20 points) Consider the following nonlinear system with parameter α.

$$\dot{x} = 1 - x - e^{-\alpha x}$$

(1) Find the critical value α_c and the fixed point x^* such that the bifurcation occurs.

(2) **Sketch** the phase portraits for $\alpha < \alpha_c$, $\alpha = \alpha_c$ and $\alpha > \alpha_c$.

(3) Classify the bifurcation.

(4) **Sketch** the bifurcation diagram.

(1) $\dot{x} = 0 = 1 - x - e^{-\alpha x}$ $\quad (\tilde{x} = 0)$

$\approx 1 - x - (1 - \alpha x + \alpha^2 x^2 + \cdots)$

$\approx \cancel{1} - x \cancel{-1} + \alpha x - \alpha^2 x^2$

$= (\alpha - 1)\tilde{x} - \alpha^2 \tilde{x}^2 \approx \alpha x - k x^2$

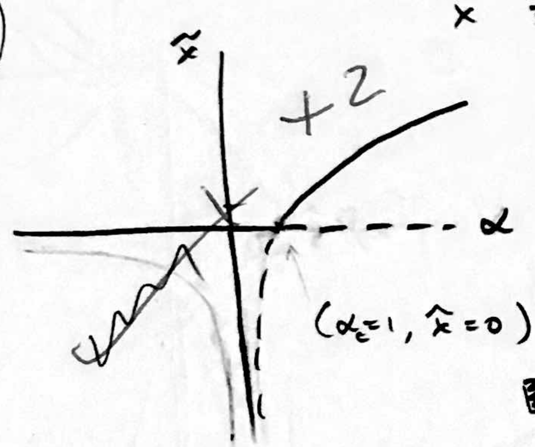
↓ Transcritical

$\Rightarrow \alpha^2 \tilde{x} = (\alpha - 1)$

$\tilde{x} = \dfrac{\alpha - 1}{\alpha^2} = \dfrac{1}{\alpha} - \dfrac{1}{\alpha^2}$

$\Rightarrow \left(\begin{array}{l} \alpha_c = 1 \\ \tilde{x} = 0 \end{array} \right)$ +4

(4)



$\dfrac{d\dot{x}}{dx} = -1 + \alpha e^{-\alpha x}$

$\left.\dfrac{d\dot{x}}{dx}\right|_{\alpha < 1}^{(\tilde{x} = 0)} = -1 + \alpha < 0$

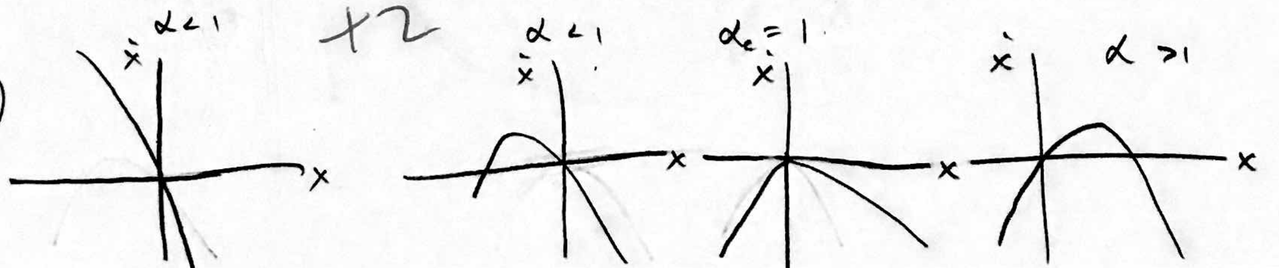
$\Rightarrow$ stable

$\left.\dfrac{d\dot{x}}{dx}\right|_{\alpha > 1}^{(\tilde{x} = 0)} = -1 + \alpha > 0$

$\Rightarrow$ unstable.

(3) → Transcritical Bifurcation

(2) +2



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4. (20 points) Consider the following nonlinear system with parameter β.

$$\dot{x} = -x + \beta \arctan(x)$$

(1) Find the critical value β_c and the fixed point x^* such that the bifurcation occurs.

(2) **Sketch** the phase portraits for $\beta < \beta_c$, $\beta = \beta_c$ and $\beta > \beta_c$.

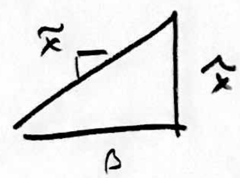
(3) Classify the bifurcation.

(4) **Sketch** the bifurcation diagram.

(1) $\dot{x} = 0 = -\tilde{x} + \beta \tan^{-1}(\tilde{x})$

$\Rightarrow \beta \tan^{-1}(\tilde{x}) = \tilde{x}$

$\Rightarrow \tan\left(\frac{\tilde{x}}{\beta}\right) = \tilde{x}$



$\frac{d}{dx}\tilde{x} = \frac{d}{dx}\tan\left(\frac{\tilde{x}}{\beta}\right)$

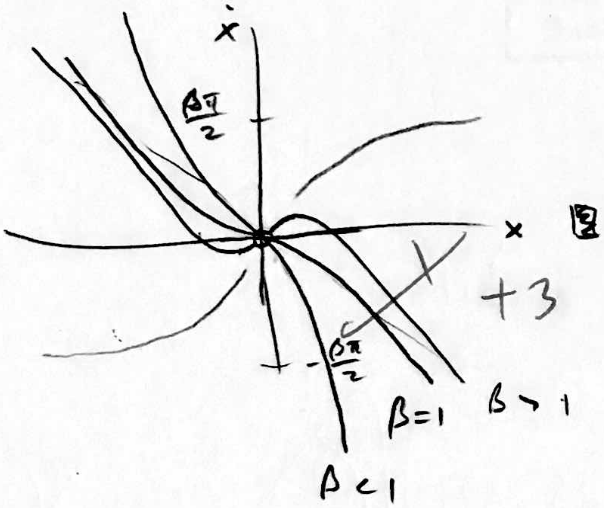
$\Rightarrow 1 = \frac{1}{\cos^2\left(\frac{\tilde{x}}{\beta}\right)} \cdot \frac{1}{\beta}$

+4

$\Rightarrow \beta_c = 1$

$\tilde{x} = 0$

(2)

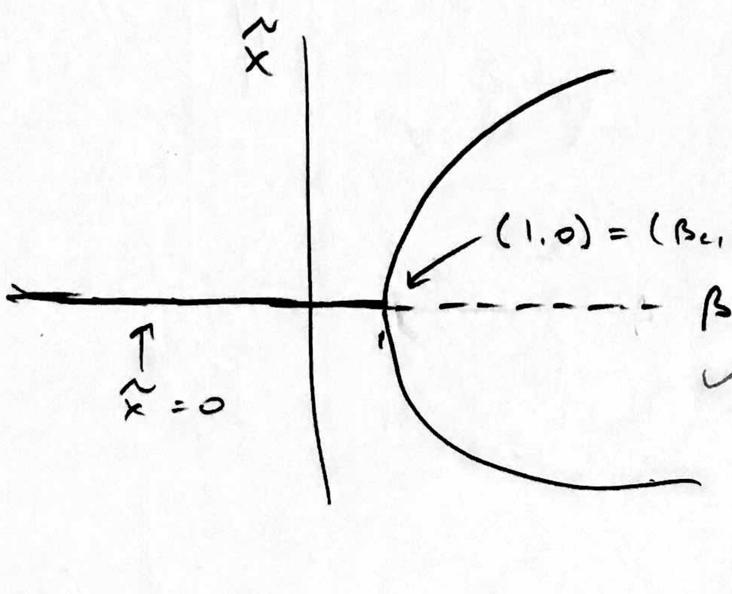


+3

(3) $\Rightarrow$ Supercritical Pitchfork Bifurcation

+2

(4)



+5