Spring 2022

Math 134 - Linear and Nonlinear Systems of Differential Equations

Midterm Exam

Name: [redacted]

UID: [redacted]

Important information regarding the exam:

1. Please write your name and student ID number wherever indicated.
2. **Please write legibly and clearly, and within the margins of the page.** Write as if you are going to grade!
3. You may not use calculators, books, notes, or any other material to help you.
4. **You must show your work to receive full credit, unless stated explicitly otherwise.**
5. The maximum score that you can earn is 54 points. 4 points are bonus.

Question	Points	Score
I	6	
II	8	
III	6	
IV	15	
V	15	
Total	50	

Problem I (3 + 3 points)

State true or False with justification.

a. Since all vector fields on the line are gradient systems, therefore it must be true that vector fields defined on the circle must also be gradient systems.

[redacted]

b. $\dot{\theta} = \omega\theta$, for $\omega > 0$, defines a vector field on the circle.

[redacted]

Problem II (8 points)

Consider the following system,

$$\dot{x} = 1 + x^2 + 3\sin^2(x), \qquad x(0) = 1.$$

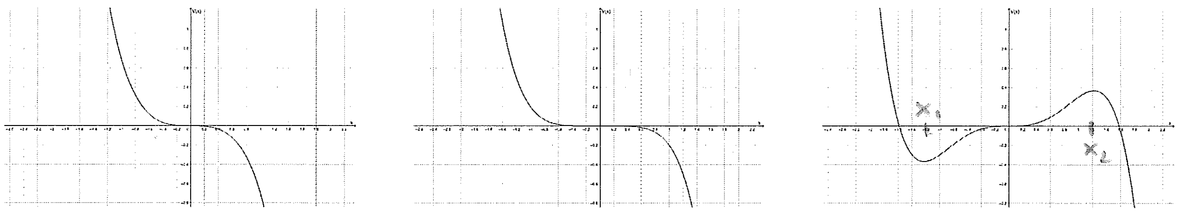
[redacted]

Show that there exists a finite T such that the solution blows up as $t \to T$.
Hint: Compare the solutions to those of $\dot{x} = 1 + x^2$.

[redacted]

Problem III (6 points)

Consider the system $\dot{x} = f(x, r)$. A potential function for f has the following qualitative graphs for different values of r.



(a) $r < 0$ (b) $r = 0$ (c) $r > 0$

Sketch the bifurcation diagram. Identify the stability of each branch clearly.

[redacted]

[redacted]

Name: ______________

Problem IV (10 + 2 + 3 = 15 points)

Consider the system:

$$\dot{x} = x + \frac{rx}{1+x^2} = f(x)$$

[redacted]

1. Sketch all the qualitatively different phase portraits as r varies. On your phase portrait, indicate the direction of the flow, find all fixed points, and classify their stability.
2. Determine the value of r at which a bifurcation occurs. Classify the type of bifurcation.
3. Sketch the bifurcation diagram. Label the axes. Find explicit formulas for the curves that you draw and label the curves with these formulas.

[redacted]

3.

[redacted]

Name: ______________

Problem V (6 + 6 + 3 = 15 points)

The equation $dN/dt = rN(1 - N/K) - H$ provides an extremely simple model of a fishery for $r, K, H > 0$. In the absence of fishing, the population is assumed to grow logistically. The effects of fishing are modeled by the term $-H$, which says that fish are caught or "harvested" at a constant rate H, independent of their population N. (This assumes that the fishermen aren't worried about fishing the population dry–they simply catch the same number of fish every day.)

(a) Show that the system can be rewritten in dimensionless form as

$$\frac{dx}{d\tau} = x(1 - x) - h,$$

for suitably defined dimensionless quantities x, τ, h.

(b) Sketch all qualitatively different phase portraits as h varies. On your phase portrait, indicate the direction of the flow, find all fixed points, and classify their stability.

(c) Show that a bifurcation occurs at a certain value h_c and x^*, and classify this bifurcation. *Bonus:* Prove analytically that bifurcation takes place at the said point and that it is of the type you claim. (4 points)

[redacted]