Math 132H, Complex Analysis (Honors), Midterm 2

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Name: Jacob Zhang

Student ID number: 605 125 539

Instructions

- Put down your name and UID above.
- You have 50 minutes to complete this exam. There are 4 problems, worth a total of 48 points.
- This test is closed book and closed notes. No cheat sheets, notes, books, calculators, cell phones, laptops, or any other references or electronic devices are allowed.
- For full credit, show all of your work legibly. Points will not be given to answers without proper justification. Please write your solutions in the space below the questions; indicate if you go over the page and/or use scrap paper.
- No cheating! Cheating of any kind, once confirmed, will invalidate the entire exam.
- Please do not remove the staple or detach this cover page.

Question:	1	2	3	4	Total
Points:	12	12	12	12	48
Score:	12	12	5	12	41

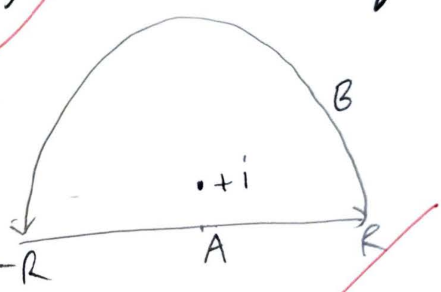
Problem 1. *12 points*

Evaluate the integral

$$\int_{\mathbb{R}} \frac{x^2 \cos x}{(1+x^2)^2}\, dx.$$

Let $f(z) = \dfrac{z^2 e^{iz}}{(1+z^2)^2}$ and consider $\int_\gamma f(z)\,dz$,

where γ is the path consisting of a line A from $-R$ to R and then the upper semicircle B from R to $-R$.



We have

$$\int_\gamma f(z)\,dz = 2\pi i \operatorname{Res}_{z=i} f(z)$$

since i is the only singularity of f in the upper half-plane.

Now

$$\lim_{R\to\infty} \int_\gamma f(z)\,dz = \lim_{R\to\infty} \int_A f(x)\,dx + \lim_{R\to\infty} \int_B f(z)\,dz$$

and

$$\left| \int_B f(z)\,dz \right| \leq \operatorname{len} B \cdot \sup_{z\in B} |f(z)| = \pi R \cdot \sup_{z\in B} \frac{z^2}{(1+z^2)^2} \cdot [e^{iz}]$$

$$\leq \pi R \cdot \frac{R^2}{(R^2-1)^2} \cdot 1 = \frac{\pi R^3}{(R^2-1)^2} \xrightarrow[R\to\infty]{} 0$$

since $z = a+bi$ with $b \geq 0 \to |e^{iz}| = |e^{ai-b}| = |e^{-b} e^{ai}| = e^{-b} \leq 1$.

Therefore

$$\operatorname{Re}\left(\lim_{R\to\infty} \int_\gamma f(z)\,dz \right) = \operatorname{Re}\left(\lim_{R\to\infty} \int_A f(z)\,dz \right) = \int_{-\infty}^{\infty} \frac{x^2 \cos x}{(1+x^2)^2}\,dx.$$

Problem 2. *12 points*

Let $f(z)$ be an entire function. Denote $M(r) = \sup_{|z| \le r} |f(z)|$. Suppose f satisfies $M(2r) \le 6M(r)$ for all $r > 0$. Show that f is a polynomial of degree not exceeding 2.

~~Using that the power series of f at 0 converges everywhere (since f entire) write~~

~~$$f(z) = a_0 + a_1 z + a_2 z^2 + z^3 g(z)$$~~

~~where $g(z)$ is also entire.~~

We show that $f^{(3)}(z) \equiv 0$ on $\mathbb{C}$, which implies f is a degree 2 polynomial. Fix $z_0 \in \mathbb{C}$. Then by the Cauchy formula

$$f^{(3)}(z_0) = \frac{3!}{2\pi i} \int_{C_r(z_0)} \frac{f(z)\,dz}{(z-z_0)^4}$$

3

and so for all r,

$$|f^{(3)}(z_0)| \le \left| \frac{3!}{2\pi i} \int_{C_r(z_0)} \frac{f(z)\,dz}{(z-z_0)^4} \right| \le \frac{3}{\pi} \left|\operatorname{len} C_r\right| \sup_{z \in C_r(z_0)} \left| \frac{f(z)}{(z-z_0)^4} \right|.$$

$$\le \frac{3}{\pi} \cdot \frac{2\pi r \cdot M(r+|z_0|)}{r^4} = \frac{6}{r^3} M(r+|z_0|).$$

2

since $C_r(z_0) \subset \overline{D}_{r+|z_0|}(0)$.

Now using the given condition and some induction we have

$$M(2^n |z_0|) \le 6^n M(|z_0|).$$

Plugging in $r = (2^n - 1)|z_0|$ and letting $n \to \infty$, we have

4

$$|f^{(3)}(z_0)| \le \frac{6}{(2^n-1)^3} M(2^n |z_0|) \le \frac{6 \cdot 6^n}{8^n} M(|z_0|)$$

$$\le \left(\tfrac{3}{4}\right)^n \cdot 6\, M(|z_0|) \to 0.$$

Therefore since z_0 arbitrary, $f^{(3)}(z_0) = 0 \;\forall z_0 \in \mathbb{C}$, as desired.

Problem 3. *12 points*

Find all possible entire functions $f(z)$ such that $f(f(z)) = z$ for all $z \in \mathbb{C}$. Justify your answer.

We claim the only such functions are $f(z) = z$ and $f(z) = c - z$.* Indeed, suppose that $f(f(z)) = z$ ~~for all $z \in \mathbb{C}$. Consider~~ Then — Noted

$$\frac{d}{dz}\left(f(f(z)) \right) = f'(z)\, f'(f(z)) = 1 \quad \text{for all } z.$$

2' 2' 1'

Now $f'(z)$ is an entire function, which can never take the value 0. Therefore $f'(z) = a e^{g(z)}$, where $g(z)$ is entire (this fact is proved in Stein and Shakarchi). and $g(0)=0$

So $a e^{g(z)} \cdot a e^{g(f(z))} = a^2 e^{g(z) - g(f(z))} = 1$,

which implies that $g(z) - g(f(z)) = 0$ for all z.

Therefore $g(z) = g(f(z))$ for all z. But since

* We at least verify that f linear $\to f(z) = z,\ c - z$.

$$a(az+b)+b = z \to a^2 z + (ab+b) = z$$

3'

$$\to \begin{array}{l} a^2 = 1 \\ b(a+1) = 0 \end{array}$$

$$\to a = 1 \text{ and } b = 0 \text{ or } a = -1.$$

Problem 4.

(a) [6 points] Suppose $f(z)$ and $g(z)$ are meromorphic functions on $\mathbb{C}$, satisfying that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Show that $f(z) = ag(z)$ for some $a \in \mathbb{C}$ with $|a| \leq 1$.

(b) [6 points] Suppose $f(z)$ is a meromorphic function on $\mathbb{C}$, with at least one pole. Let $h(z) = e^{1/f(z)}$. Is $h(z)$ meromorphic? If yes, give a proof. If not, find a counter-example.

a. Let $h(z) = \frac{f(z)}{g(z)}$. Clearly, h is meromorphic on $\mathbb{C}$. Now outside of the set where $g(z) = 0$, we have $|h(z)| = \frac{|f(z)|}{|g(z)|} \leq \frac{|g(z)|}{|g(z)|} = 1$. Therefore h is bounded in a neighborhood of each one of its singularities, so each singularity is removable. It follows that $h(z)$ can be extended to an entire function $h: \mathbb{C} \to \mathbb{C}$. But since $|h(z)| \leq 1$ for all z by continuity, h is a bounded entire function, and therefore $h(z) = a$ ($|a| \leq 1$) for all z, so $f(z) = ag(z)$ for all z as desired.

b. Let $f(z) = \frac{z}{z+1}$. $f(z)$ has a pole of order 1 at -1.

However, $e^{\frac{1}{f(z)}} = e^{\frac{z+1}{z}} = e^{1+\frac{1}{z}} = e \cdot e^{\frac{1}{z}}$ is not meromorphic, since $e^{\frac{1}{z}}$ has an essential singularity at 0.

So $h(z)$ is not necessarily meromorphic.