Math 132H, Complex Analysis (Honors), Midterm 1

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Instructions:

- Put down your name and UID above.
- You have 50 minutes to complete this exam. There are four questions, worth a total of 48 points.
- This test is closed book and closed notes. No cheat sheets, notes, books, calculators, cell phones, laptops, or any other references or electronic devices are allowed.
- For full credit, show all of your work legibly. Points will not be given to answers without proper justification. Please write your solutions in the space below the questions; indicate if you go over the page and/or use scrap paper.
- No cheating! Cheating of any kind, once confirmed, will invalidate the entire exam.
- Please do not remove the staple or detach this cover page.

Question:	1	2	3	4	Total
Points:	12	12	12	12	48
Score:	6	~~0~~	~~4~~	11	~~35~~

8 12 37

Problem 1.

Let $f(z) = u(x, y) + iv(x, y)$, where u and v are real and imaginary parts of f, respectively. u and v are continuously differentiable. Let $z_0 = x_0 + iy_0 \in \mathbb{C}$. Suppose the limit

$$\lim_{z \to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} \right|$$

exists.

(a) [6 points] Show that at (x_0, y_0),

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \quad \text{and} \quad \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0.$$

(b) [6 points] Use the result above to show that either $f(z)$ or $\overline{f(z)}$ is holomorphic at z_0.

a. Writing this out, we have that

$$\lim_{(x,y)\to(x_0,y_0)} \left| \frac{u(x,y) + iv(x,y) - u(x_0,y_0) + iv(x_0,y_0)}{(x+iy) - (x_0+iy_0)} \right|$$

exists; suppose it is equal to L.

Letting $y = y_0$ and allowing $x \to x_0$, we obtain

$$L = \lim_{x\to 0} \left| \frac{u(x,y_0) - u(x_0,y_0)}{x - x_0} + i\frac{v(x,y_0) - v(x_0,y_0)}{x - x_0} \right|$$

$$= \left| \lim_{x \to 0} \left[\frac{u(x,y_0) - u(x_0,y_0)}{x - x_0} + \frac{iv(x,y_0) - v(x_0,y_0)}{x - x_0} \right] \right| = \left| \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \right| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}$$

+6

On the other hand, letting $x = x_0$ and taking the limit as $y \to y_0$, we obtain $L = \sqrt{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2}$ by an identical argument. Therefore $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$.

On the other hand, if we let $x = x_0 + t$, $y = y_0 + t$, and take the limit as $t \to 0$, then

$$L = \lim_{t \to 0} \left| \frac{u(x_0+t, y_0+t) - u(x_0,y_0)}{(1+i)t} + i\frac{v(x_0+t, y_0+t) - v(x_0,y_0)}{(1+i)t} \right|$$

$$= \left| \lim_{t\to 0} \frac{u(x_0+t, y_0+t) - u(x_0, y_0+t) + u(x_0, y_0+t) - u(x_0,y_0)}{(1+i)t} + \frac{i(\dots)}{(1+i)t} \right|$$

$$= \left| \frac{1}{1+i} \left[\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\right) \right] \right| = \sqrt{\frac{1}{2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right)}$$

but since $L = \sqrt{\frac{1}{2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right)}$ already, $\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0$ necessarily. (Part b on back).

Problem 2. *12 points*

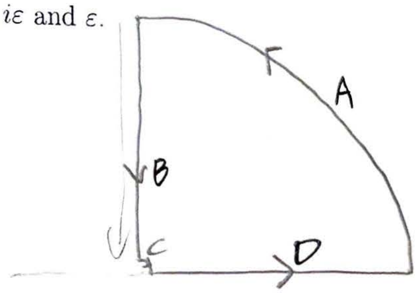
Evaluate the integral

$$\int_0^{+\infty} \frac{\cos x - e^{-x}}{x}\,dx. \quad = \operatorname{Re} I$$

Hint: You may consider

$$f(z) = \frac{e^{iz} - e^{-z}}{z} \qquad -\frac{z^2}{2} \quad -\frac{z^2}{2}$$

and a contour consisting of the following four parts: the straight line from ε to R, the circular arc centered at 0 connecting R and iR, the straight line from iR to $i\varepsilon$, and the circular arc centered at 0 connecting $i\varepsilon$ and ε.



Let I = the desired value, $\int_0^{\infty} f(z)\,dz$.

We compute $\int_{ABCD} f(z)\,dz = \int_A f\,dz + \int_B f\,dz + \int_C f\,dz + \int_D f\,dz = 0$ by Cauchy's theorem, since f is holomorphic on the domain, being a product/sum/composition of holomorphic functions.

On the other hand,

$$\lim_{\substack{n\to\infty \\ R=(2n+\frac{1}{2}\pi)}} \int_A f\,dz \le \left|\sup_{z\in A} |f(z)|\right| \cdot \operatorname{len}(A) \text{ which goes to } 0.$$

why ? wrong -3

$$\lim_{\substack{\epsilon\to 0 \\ R\to\infty}} \int_B f\,dz = \int_{+\infty}^{0} f(it)\cdot i\,dt = \int_{+\infty}^{0} \frac{e^{-t} - e^{-it}}{it}\,\cancel{i}\,dt$$

?

$$= \int_0^{+\infty} \frac{e^{it} - e^{-t}}{t}\,dt = I$$

$$\left|\lim_{\epsilon\to 0} \int_C f\,dz\right| \le \lim \sup_{|z|=\epsilon} |f(z)| \cdot \operatorname{len} C = \lim_{\epsilon\to 0} (\text{bounded})\cdot\epsilon \to 0 \quad -1$$

as $\lim_{z\to 0} f(z) = 1+i$ or something

$$\lim_{\substack{\epsilon\to 0 \\ R\to\infty}} \int_D f\,dz = I \text{ by definition.}$$

therefore $2I + 0 + 0 = 0$, so $I = 0$.

Problem 3.

Consider the following power series

$$f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(k+1)^k} z^k, \quad g(z) = \sum_{k=1}^{\infty} k^{-1} z^{k!}.$$

(a) [5 points] Find out the radii of convergence of f and g.

Hint: Stirling's formula: $n! \approx \sqrt{2\pi n}\, n^n e^{-n}$.

(b) [4 points] Show that f is divergent at all points on the boundary of its disc of convergence.

(c) [3 points] Show that g is divergent at infinitely many points on the boundary of its disc of convergence.

a. The radius of convergence of ~~f~~ is the inverse of $\lim_{k\to\infty} \left|\frac{(-1)^k k!}{(k+1)^k}\right|^{1/k} = \lim_{k\to\infty} \frac{1}{k+1} (|k!|)^{1/k}$

$$= \lim_{k\to\infty} \frac{k}{k+1} e^{-1} k^{-\frac{1}{2k}}$$

$$= e^{-1}$$

So radius of convergence of f is $\boxed{e}$.

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On the other hand the radius of convergence of g is $\boxed{1}$ since if $|z|<1$ we have $|\sum k^{-1} z^{k!}| < \sum |k^{-1}||z|^k$ converges absolutely, but if $|z| = 1$ then $g(1) = \sum_{k=1}^{\infty} k^{-1}$ diverges.

b. If $|z| = e$, then $\lim_{k\to\infty} \left|\frac{(-1)^k k!}{(k+1)^k} z^k\right| = \lim_{k\to\infty} e^k \left|\frac{k!}{k^k}\right| \left|\frac{k^k}{(k+1)^k}\right|$

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$$= \left(\lim_{k\to\infty} \sqrt{2\pi k}\right) \cdot e^{-1} \to \infty,$$ so the series diverges

c. If $z = e^{\frac{2\pi i}{N}}$, then we claim $g(z)$ diverges.

Indeed, the series will contain the tail

$$\sum_{k=N}^{\infty} k^{-1} e^{\frac{2\pi i}{N} \cdot k!} = \sum_{k\geq N} k^{-1} \cdot 1 = \sum_{k \geq N} k^{-1}$$ which diverges.

since $\frac{k!}{N} \in \mathbb{Z}$.

Problem 4.

(a) [6 points] Prove that a sequence of complex numbers $z_n \to z$ if and only if

$$|\mathrm{Re}\,(z_n - z)| + |\mathrm{Im}\,(z_n - z)| \to 0.$$

(b) [6 points] Sketch the image of

$$\{z \in \mathbb{C} : 0 \le \mathrm{Im}\, z \le |\mathrm{Re}\, z|\}$$

under the mapping e^z. Your answer should come with necessary explanation. Be as precise as you can.

a. By definition, $z_n \to z$ if and only if

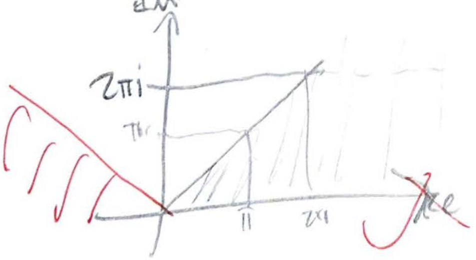
$$|z_n - z| = \sqrt{(\mathrm{Re}(z_n-z))^2 + \mathrm{Im}(z_n-z)^2} \to 0.$$

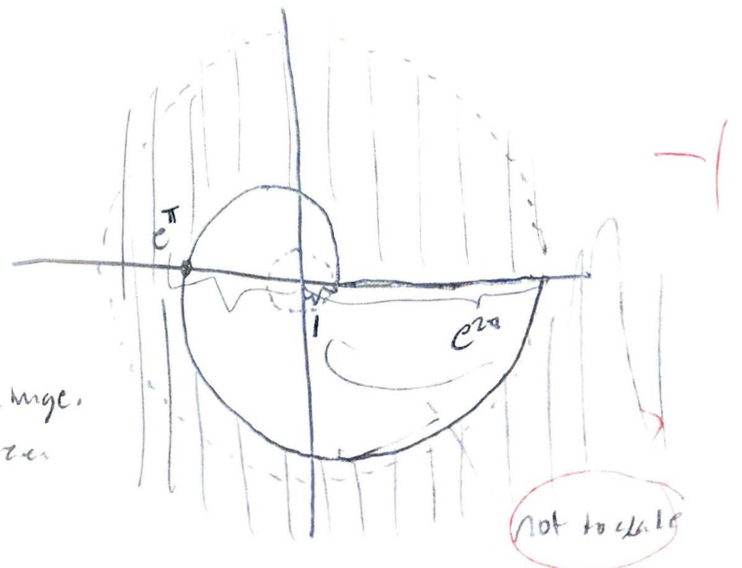
which occurs if and only if

$$(\mathrm{Re}(z_n-z))^2 + (\mathrm{Im}(z_n-z))^2 \to 0.$$

Now suppose that for all $n > N$, $(\mathrm{Re}(z_n-z))^2 + \mathrm{Im}(z_n-z)^2 < \varepsilon$. This implies that $|\mathrm{Re}(z_n-z)| < \sqrt{\varepsilon}$, $|\mathrm{Im}(z_n-z)| < \sqrt{\varepsilon}$, and so $|\mathrm{Re}(z_n-z)| + |\mathrm{Im}(z_n-z)| < 2\sqrt{\varepsilon}$. On the other hand, if $\forall n \ge N$ $|\mathrm{Re}(z_n-z)| + |\mathrm{Im}(z_n-z)| < \varepsilon$, then similarly $\mathrm{Re}(z_n-z)^2 + \mathrm{Im}(z_n-z)^2 < 2\varepsilon^2$. Therefore $(\mathrm{Re}\, z_n - z)^2 + (\mathrm{Im}\, z_n - z)^2 \to 0$ iff $|\mathrm{Re}\, z_n - z| + |\mathrm{Im}\, z_n - z| \to 0$ as desired.

b.





All points such that $|w| \ge e^{2\pi}$ are in the image.
No points such that $|w| < 1$ are in [illegible]