Question (1) (20 Points). Let $f(z)$ be the complex valued function of the complex variable $z = x + iy$ that is given by the expression

$$f(z) = |x^2 - y^2| + 2ixy$$

valid for all $z \in \mathbb{C}$. Find, precisely, the region of $\mathbb{C}$ in which f is *analytic.*

let $u = |x^2 - y^2|$ and $v = 2xy$

① u and v has to have first derivative and be continuous

so $x^2 \neq y^2$

② if $x^2 > y^2$

$u = x^2 - y^2$, $v = 2xy$

$u_x = 2x = v_y$

$u_y = -2y = -v_x$ ✓

if $x^2 < y^2$

$u = y^2 - x^2$, $v = 2xy$

$u_x = -2x \neq v_y$ X

so the region of C is any place in $\mathbb{C}$ where

$x^2 > y^2$ and $x \neq 0, y \neq 0$

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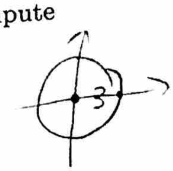
Question (2) (20 Points). Let $\alpha \in \mathbb{R}$ denote a real parameter and consider the function

$$f(z) = z e^{i\alpha z^2}.$$

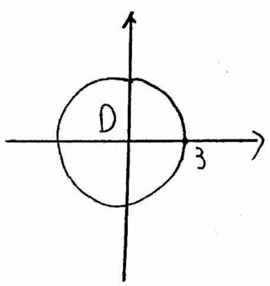
Let Γ denote the circular contour of radius 3 centered at the origin, i.e. $\Gamma : |z| = 3$. Compute

$$\oint_\Gamma \frac{f(z)}{(z - [1+i])} dz.$$



Justify all steps.

$f(z) = z e^{i\alpha z^2}$



f(z) is analytic in the region D included Γ

$\oint_\Gamma \frac{f(z)}{(z-[1+i])} dz = 2\pi i f(z_0)$, $z_0 = 1+i$ use Cauchy Integral

$f(z_0) = (1+i) e^{i\alpha(1+i)^2} = (1+i) e^{i\alpha(2i)} = (1+i) e^{-2\alpha}$

so $2\pi i f(z_0) = 2\pi i (1+i) e^{-2\alpha}$

$= 2\pi e^{-2\alpha} (i-1)$

so $\oint_\Gamma \frac{f(z)}{(z-[1+i])} dz = 2\pi e^{-2\alpha} (i-1)$

Question (3) (20 Points). Let $\beta \in \mathbb{R}$ satisfy $|\beta| > 1$.

Part A (10 Points). Show that

$$\operatorname{Re}\left[\frac{1}{\beta + e^{i\theta}}\right] = \frac{\beta + \cos\theta}{\beta^2 + 2\beta\cos\theta + 1}$$

where θ is the usual "angle parameter".

$$\frac{1}{\beta + e^{i\theta}} = \frac{1}{(\beta + \cos\theta) + i\sin\theta} = \frac{(\beta + \cos\theta) - i\sin\theta}{(\beta + \cos\theta)^2 + \sin^2\theta} = \frac{(\beta + \cos\theta) - i\sin\theta}{\beta^2 + 2\beta\cos\theta + 1}$$

so $\operatorname{Re}\left[\frac{1}{\beta + e^{i\theta}}\right] = \frac{\beta + \cos\theta}{\beta^2 + 2\beta\cos\theta + 1}$

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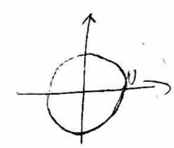
Part B (10 Points). Using your result from Part A of this problem compute, quickly and easily,

$$\int_0^{2\pi} \frac{\beta + \cos\theta}{\beta^2 + 2\beta\cos\theta + 1}\, d\theta.$$



$$\int_0^{2\pi} \frac{\beta + \cos\theta}{\beta^2 + 2\beta\cos\theta + 1}\, d\theta = \int_0^{2\pi} \operatorname{Re}\left[\frac{1}{\beta + e^{i\theta}}\right] d\theta = \operatorname{Re} \int_0^{2\pi} \frac{1}{\beta + e^{i\theta}}\, d\theta$$

$= \cos\theta + i\sin\theta$

let $z = \cos\theta + i\sin\theta$, $dz = (-\sin\theta + i\cos\theta)d\theta = iz\, d\theta$

$$= \operatorname{Re} \oint_{|z|=1} \frac{1}{z + \beta} \frac{dz}{(iz)} = \operatorname{Re} \oint \frac{-i}{z(z+\beta)} = \oint \frac{-i/(z+\beta)}{z}$$

on a circle Γ include origin

use Cauchy Integral, because $\beta > 1$

$$\oint \frac{-i/(z+\beta)}{z} = 2\pi i \frac{-i}{(0+\beta)} = \frac{2\pi i(-i)}{\beta} = \frac{2\pi}{\beta}$$

so $\int_0^{2\pi} \frac{\beta + \cos\theta}{\beta^2 + 2\beta\cos\theta + 1}\, d\theta = \frac{2\pi}{\beta}$

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Question (4) (25 Points). Let $L > 0$ denote a real number and consider the contour Γ_L which consists of two pieces, $\Gamma_L = \Gamma_{L_1} \cup \Gamma_{L_2}$, both of which are strait lines. The line Γ_{L_1} starts at the point $z = 1$ on the real axis and connects to the point $z = iL$ on the imaginary axis. The line Γ_{L_2} stats at $z = iL$ and goes strait down the imagionary axis till it hits the origin. Thus, in particular, Γ_L consists of two legs of a right triangle.

Part A (5 Points). Write a parametric description for the lines (contours) Γ_{L_1} and Γ_{L_2}.

$\Gamma_{L_1} : 1 + t(iL-1), \quad 0 \le t \le 1$

$\Gamma_{L_2} : iL + t(-iL), \quad 0 \le t \le 1$

Part B (20 Points). Let $f(z) = (\cos z)e^{-z^2}$. Show, regardless of the value of L (even if L is "large") that

$$\left| \int_{\Gamma_L} f(z)dz \right| \le 1$$

$x+iy$

$$\frac{e^{-iz} + e^{iz}}{2} = \frac{e^{x+iy} + e^{x-iy}}{2}$$

e^x

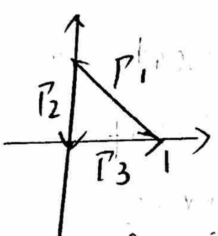
Hint: Consider the case $L \gg 1$ and draw a picture.

⋆ Also (small) bonus for a better estimate than the above. But only do this if you have time to spare.⋆

$1 - 2x = 0$, $x = \frac{1}{2}$, $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$f(z) = \cos z \, e^{-z^2}$ is analytic

so $\int_{\Gamma_1 + \Gamma_2 + \Gamma_3} f(z)\, dz = 0$



so $\int_{\Gamma_L} f(z)dz + \int_{\Gamma_3} f(z)dz = 0 \Rightarrow \int_{\Gamma_L} f(z)dz = -\int_{\Gamma_3} f(z)dz$

for $\left|\int_{\Gamma_3} f(z)dz\right|$ $\quad |\Gamma_L| = 1$

$|f(z)| = |\cos z||e^{-z^2}|$

$|\cos z| \le |e^x, \quad |e^{-z^2}| = |e^{-(x^2-y^2+2ixy)}| = e^{-(x^2-y^2)}$

$|f(z)| = e^{x-(x^2-y^2)}$ in Γ_3, $y = 0$ so $|f(z)| = e^{x-x^2} \le e^{\frac{1}{4}}$

so $\left|\int_{\Gamma_L} f(z)dz\right| = \left|\int_{\Gamma_3} f(z)dz\right| \le |f(z)||\Gamma_{L_3}| \le e^{\frac{1}{4}}$

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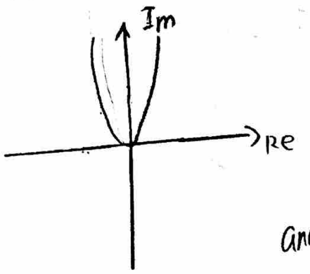
Question (5) (20 Points). A function $f(z)$ called $z^{1/4}$ has the following four properties:

I) It agrees with the usual real $x^{1/4}$ on the positive real axis.
II) It is discontinuous on the positive imaginary axis (where the function is not defined).
III) Everywhere except for the origin (and the positive imaginary axis, where it is not defined) f is analytic.
IV) Everywhere except for the positive imaginary axis (where the function is not defined) it satisfies $f^4 = z$.

Find an explicit formula for such a function e.g., using polar coordinates, and evaluate $\underline{f(-4)}$ expressing your answer in the form $a + ib$ with a and b real.

First, we have $f(z) = r^{\frac{1}{4}} e^{i\frac{\theta + 2m\pi}{4}}$

$\theta \in [-\frac{3}{2}\pi + 2m\pi, \frac{1}{2}\pi + 2m\pi)$ $m = 0, 1, 2, 3$



when on the positive real axis, angle should be $0 + 2m\pi$ so that $f(z) = x^{\frac{1}{4}}$, let $\varphi = \frac{\theta + 2k\pi}{4}$

$$\frac{\theta + 2m\pi}{4} = 2m\pi$$

$$\theta = 6m\pi$$

only if $m = 0$ is possible

so $\theta \in [-\frac{3}{2}\pi, \frac{1}{2}\pi)$ because $m = 0$ is the only possibility

so $f(z) = r^{\frac{1}{4}} e^{i\frac{\theta}{4}}$, $\theta \in [-\frac{3}{2}\pi, \frac{1}{2}\pi)$

$f(-4) = 4^{\frac{1}{4}} e^{\frac{-i\pi}{4}} = -4^{\frac{1}{4}}$

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