1	4.5
2	3
3	3.5
4	5
5	0
T	16

MATH 132 Midterm, Fall 2016

Name: John Stucky

Justify All Your Answers. No Points Will Be Given Without Sufficient Reasoning/Calculations.

Problem 1. (5)

(I) Find the derivative of $w = (z+1)^{\frac{1}{n}}$ in two ways by (i) considering w as the inverse function of $z = w^n - 1$ and using inverse function theorem or (ii) using the formula $w = e^{\frac{1}{n} \cdot log(z+1)}$.

In both methods, specify the branches of the functions used in your computation.

(II) Are the results from the two methods the same? Are the derivatives of different branches the same?

(I)(i) $z(w) = w^n - 1$ $\quad D = \mathbb{C} \setminus (-\infty, -1]$

$dz = nw^{n-1} dw$

$$\frac{dw}{dz} = \frac{1}{\frac{dz}{dw}} = \frac{1}{nw^{n-1}} = \frac{1}{n((z+1)^{\frac{1}{n}})^{n-1}} = \frac{1}{n(z+1)^{\frac{n-1}{n}}} = \frac{1}{n\frac{(z+1)}{(z+1)^{1/n}}}$$

$$= (z+1)^{1/n} \frac{1}{n(z+1)}$$

(principal branch)

(ii) $w = e^{\frac{1}{n}\log(z+1)}$ $\quad D = \mathbb{C} \setminus (-\infty, -1]$

$dw = e^{\frac{1}{n}\log(z+1)} \frac{1}{z+1}$ (principal log branch)

$= (z+1)^{\frac{1}{n}} \frac{1}{n} \frac{1}{z+1}$

(II) Yes, the results from the two methods are the same. At different branches, the derivatives are not the same. how? $-.5$

$$f(z) = \frac{az+b}{cz+d}$$

Problem 2. (5)

(I)Find the fractional linear transformation f such that $f(0) = 1+i$, $f(1) = 2i$, $f(\infty) = -1+i$.

(II) Find the images of the x-axis and y-axis of f.

$f(0) = \frac{b}{d} = 1+i \Rightarrow b = (1+i)d$

$f(\infty) = \frac{a}{c} = -1+i \quad a = (-1+i)c$

$f(1) = \frac{a+b}{c+d} = 2i \Rightarrow \quad a+b = 2ic+2id$

$(-1+i)c + (1+i)d = 2ic+2id$

$-c+ic+d+id = 2ic+2id$

$-c-ic = id-d$

$-c(1+i) = d(i-1)$

$-c(1+i)^2 = 2d$

$-c(1+2i-1) = 2d$

$-2ic = 2d$

$c = \frac{-1}{i}d$

$c = id$

$a = (-1+i)c = (-1+i)id = (-i-1)d$

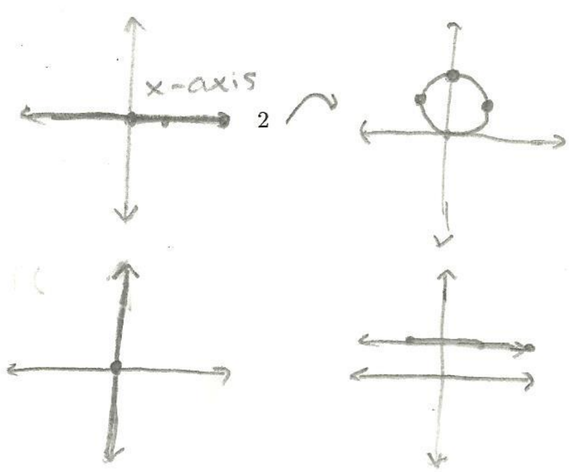
$$f(z) = \frac{az+b}{cz+d} = \frac{(-1-i)dz+(1+i)d}{idz+d} = \frac{(-1-i)z+(1+i)}{iz+1}$$

$$\boxed{f(z) = \frac{(-1-i)z+1+i}{iz+1}}$$

$z=1 \mapsto 0$ X

-2

(II)



x-axis maps to unit disk centered at i.

$f(i\infty) = \frac{a}{c} = -1+i$

$f(i) = \frac{-i+1+1+i}{0} = \infty$

y-axis maps to line $Im(z)=1$.

Problem 3. (5)

(I) Assume that $f(z) = u(x,y) + v(x,y)i$ is a analytic function defined on a domain D such that the imaginary part $v(x,y)$ is a constant. Show that f is a constant.

(II) Prove or disprove the following statements (a) both real and imaginary parts of $f(z) = \bar{z}^3$ are harmonic on the complex plane **C**; (b) $f(z) = \bar{z}^3$ is analytic on **C**.

(I) f is analytic

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -$$

v is const $\Rightarrow \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$

$\Rightarrow \frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial y} = 0$

$$u = \int \frac{\partial u}{\partial x} dx + g(y) = \int \frac{\partial u}{\partial y} + h(x)$$

$$= 0 + g(y) = 0 + h(x)$$

$\Rightarrow$ u = const

u is const and v is const

$\Rightarrow$ f is constant

(II) (a) $\bar{z}^3 = (x - yi)^3 = x^3 - 3x^2yi - 3xy^2 + iy^3$

$$= x^3 - 3xy^2 + i(y^3 - 3x^2y)$$

$u = x^3 - 3xy^2$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial u}{\partial y} = -6xy$$

$$\frac{\partial^2 u}{\partial x^2} = 6x \qquad \frac{\partial^2 u}{\partial y^2} = -6x$$

$\Delta u = \frac{\partial^2 u}{\partial x} + \frac{\partial^2 u}{\partial^2 y} = 6x - 6x = 0 \Rightarrow$ u is harmonic ✓

$$\frac{\partial v}{\partial x} = -6xy \qquad \frac{\partial v}{\partial y} = 3y^2 - 3x^2$$

$$\frac{\partial^2 v}{\partial x^2} = -6y \qquad \frac{\partial^2 v}{\partial y^2} = 6y \qquad \Delta v = -6y + 6y = 0$$

Since $\Delta v = 0$, v is harmonic ✓

Problem 4. (5)

(i) Show that $u(x, y) = x^2y - \frac{1}{3}y^3 + 2x + 3y$ is harmonic.

(ii) Find the harmonic conjugate of u in (i) by solving the Cauchy-Riemann equation.

Note: No points will be given for (ii) without solving the Cauchy-Riemann equation.

(i) $\frac{\partial u}{\partial x} = 2xy + 2$ $\frac{\partial u}{\partial y} = x^2 - y^2 + 3$

$\frac{\partial^2 u}{\partial x^2} = 2y$ $\frac{\partial^2 u}{\partial y^2} = -2y$

$\Delta u = 2y - 2y = 0$

$\Rightarrow$ u is harmonic

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2xy + 2$

$v = \int \frac{\partial v}{\partial y} dy + g(x) = \int 2xy + 2\, dy + g(x) = xy^2 + 2y + g(x)$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$v = \int -(x^2 - y^2 + 3]\, dx + h(y)$

$= -\frac{1}{3}x^3 + xy^2 - 3x + h(y)$

$v = \boxed{-\frac{1}{3}x^3 + xy^2 + 2y - 3x} + C$

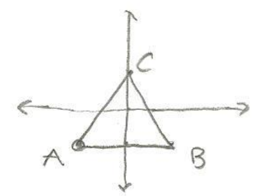
Problem 5. (5)

(i) Decide if the differentials $\phi_1 = (\frac{-y}{x^2+y^2} + x)dx + (\frac{x}{x^2+y^2} + y)dy$ and $\phi_2 = (\frac{-y}{x^2+y^2} - y)dx + (\frac{x}{x^2+y^2} + x)dy$ are closed on $\mathbf{C} \setminus 0$.

(ii) Is ϕ_1 exact on $\mathbf{C} \setminus 0$?

(iii) What is the maximal domain on which ϕ_1 is exact?

(iv) Find the line integral $\int_\gamma \phi_2$, where γ consists of the three oriented line segments AB, BC and CA with $A = (-1, -1)$, $B = (1, -1)$ and $C = (0, 1)$.



(iv) $\int_\gamma \phi_2 =$

$\gamma_1(t) = \langle t, -1 \rangle \quad -1 \le t \le 1$

$\gamma_2(t) = \langle -t, 2t-1 \rangle \quad -1 \le t \le 0$

$\gamma_3(t) = \langle$

$$\int \frac{-y}{x^2+y^2} - y\,dx + \int \frac{x}{x^2+y^2} + x\,dy$$