MATH 132: MIDTERM #3

WINTER 2014

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Instructions: Do all 6 problems starting from those you find easier to complete. Do not write (or, if you do, then erase) anything that is not relevant to the solution; all what will be found on the page will count. Each problem is worth the same amount of points. There will be little partial credit so make sure to get everything right.

DO NOT WRITE BELOW THIS LINE!

1	4	2	4	3	2
4	4	5	4	6	4

TOTAL SCORE 22

PROBLEM 1

Do as follows:

(1) Find all zeros of $f(z) = z(1 - \cos z)$ on $\mathbb{C}$ and prove why you have all of them.
(2) For each zero determine its order.

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(1)

$0 = z(1 - \cos z)$

$z = 0 \qquad 1 - \cos z = 0$

$\cos z = 1$

$\boxed{z = 2\pi k \text{ for } k \in \mathbb{Z}}$, which includes $z = 0$

$\cos z$ is periodic by 2π, so it only achieves the value 1 when z is an integer multiple of 2π.
This encompasses all zeros of $(1-\cos z)$ and also includes $z=0$, so it is all the zeros of $f(z)$. ✓

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(2)

$f'(z) = (1-\cos z) + z\sin z$

$f''(z) = \sin z + \sin z + z\cos z$

$f'''(z) = 2\cos z + \cos z - z\sin z$

$f'''(0) = 2 + 1 - 0 \neq 0$

$f'(2\pi k) = 1 - 1 + 0 = 0$ for $k \in \mathbb{Z}$

$f''(2\pi k) = 0 + 0 + 2\pi k$

$f''(0) = 0$, $f''(2\pi k \neq 0) \neq 0$

$\boxed{\text{Order 2 for } z = 2\pi k \text{ where } k = \pm 1, \pm 2, \pm 3, \ldots}$ ✓

$\boxed{\text{Order 3 for } z = 0}$ ✓

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PROBLEM 2

Consider the differential equation

$$f'(z) = \frac{\alpha + z}{z} f(z)$$

for z in an open set D. Do the following:

(1) What properties of D you need to require so that this particular equation has a solution that is analytic in D. Give the full reason for each property that you state.

(2) Choose such a D and find the explicit solution for which $f(1) = 2$.

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(1) $\boxed{\text{D must be simply connected.}}$ & domain

If f is analytic, then so is f'! D must be simply connected to integrate f'! ✓

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(2) Choose $\boxed{D = \mathbb{C} \setminus (-\infty, 0]}$

$f(1) = 2$ $\qquad f'(z) = \frac{\alpha}{z} f(z) + f(z)$

$$\int \frac{df}{f} = \int \frac{\alpha + z}{z}\, dz + C$$

$$\operatorname{Log}(f(z)) = z + \alpha \operatorname{Log}(z) + C$$

$$f(z) = e^C e^z z^\alpha = Ae^z z^\alpha = f(z)$$

$f(1) = 2 \qquad 2 = Ae2^\alpha \qquad A = 2e^{-1}2^{-\alpha} = e^{-1}2^{1-\alpha}$

$$f(z) = e^{-1}2^{1-\alpha}e^z z^\alpha = \boxed{e^{z-1}\, 2^{1-\alpha} z^\alpha}$$

not quite right but close

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PROBLEM 3

Note that $f(z) = \dfrac{\cos(z)}{\sin(z)}$ has an isolated singularity at $z = 0$. Do the following:

(1) Determine what type of isolated singularity is $z = 0$. (Justify your answer, of course.) Find the largest $r > 0$ for which $f(z)$ admits a representation as a converging Laurent series $\sum_{n\in\mathbb{Z}} a_n z^n$ on $\{z \in \mathbb{C} : 0 < |z| < r\}$.

(2) Find the coefficients a_{-2}, a_{-1}, a_0, a_1 and a_2 in this representation.

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(1) $\lim_{z\to 0} |f(z)| = \lim_{z\to 0} \left|\frac{\cos(z)}{\sin(z)}\right| = \infty$ This blows up, so it is a $\boxed{\text{pole}}$ at $z=0$ ✓

$\boxed{r = \pi}$ This is $\cot(z)$, which is undefined at $z=\pi$ ✓

you didn't quite prove this

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(2) $\mathrm{Res}(f,0) = \frac{\cos(0)}{\cos(0)} = 1$

so $\boxed{a_{-2}=0,\ a_{-1}=1,\ a_0=0,\ a_1=0,\ a_2=0}$

Need to expand, not compute residues!

PROBLEM 4

Consider the integral $\int_\Gamma \frac{z^2}{1+z^4}\,dz$ where Γ is the curve obtained by joining the endpoints of the linear segment $[-R, R]$ on the real axis by the semicircle $\{Re^{i\theta} : 0 \leq \theta \leq \pi\}$. Do the following:

(1) State, but not prove, a general theorem that will allow you to compute such integrals.

(2) Compute the integral.

Assume R is very large.

(1)

Residue theorem.

An integral $\int_\Gamma f(z)\,dz$ over a closed curve Γ equals the sum of the residues within Int Γ multiplied by $2\pi i$.

(2)

$$\int_\Gamma \frac{z^2}{1+z^4}\,dz$$

$1+z^4=0$

$z^4=-1$

$|z|^4=1$ $\quad 4\theta = \pi + 2k\pi$

$\theta = \frac{\pi}{4} + \frac{k\pi}{2}$ $\quad k = \ldots$

only z_k lie within our Γ for $k=0,1$

$$\text{Res}(f, z_0) = \frac{z^2}{4z^3} = \frac{1}{4z}$$

$$\text{Res}(f, e^{i\frac{\pi}{4}}) = \frac{1}{4}e^{-i\frac{\pi}{4}}$$

$$= \frac{1}{4}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$= \frac{1}{4\sqrt{2}}(1-i)$$

$$\text{Res}(f, e^{i(\frac{\pi}{4}+\frac{\pi}{2})}) = \frac{1}{4}e^{-i(\frac{3\pi}{4})}$$

$$= \frac{1}{4}\left(-\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$= \frac{1}{4\sqrt{2}}(-1-i)$$

$$2\pi i\left[\text{Res}(f, e^{i\frac{\pi}{4}}) + \text{Res}(f, e^{i\frac{3\pi}{4}})\right] = 2\pi i\left[\frac{1}{4\sqrt{2}}\cdot -2i\right] = \boxed{\frac{\pi}{\sqrt{2}}} = \int_\Gamma \frac{z^2}{1+z^4}\,dz$$

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PROBLEM 5

Find the Laurent series of $f(z) = \dfrac{z}{z+2}$ in

(1) the annulus $\{z \in \mathbb{C} : 0 < |z| < 2\}$
(2) the annulus $\{z \in \mathbb{C} : 2 < |z| < \infty\}$.

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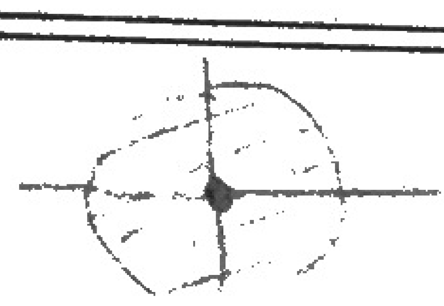
(1)



$|z| < 2 \qquad \frac{|z|}{2} < 1$

$$\frac{z}{z+2} = \frac{z}{2+z} = \frac{\frac{z}{2}}{1+\frac{z}{2}} \qquad \left|\frac{-z}{2}\right| < 1$$

$$= -\sum_{n=0}^{\infty}\left(\frac{-z}{2}\right)^{n+1} = \boxed{\sum_{n=1}^{\infty} -\left(\frac{-1}{2}\right)^{n} z^{n}}$$

✓

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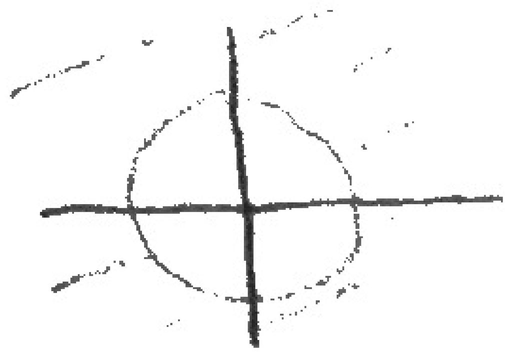
(2)



$|z| > 2 \qquad \frac{|2|}{|z|} < 1$

$$\frac{z}{z+2} = \frac{1}{1-\left(-\frac{2}{z}\right)} \qquad \left|-\frac{2}{z}\right| < 1$$

$$= \sum_{n=0}^{\infty}\left(\frac{-2}{z}\right)^{n} = \boxed{\sum_{n \in (-\infty, 0]} (-2)^{-n} z^{n}}$$

interval

4

PROBLEM 6

Suppose $f(z)$ solves

$$(1-z)f''(z) + 2zf'(z) + 8f(z) = 0$$

Assume the solution takes the form $f(z) = \sum_{n\geq 0} a_n z^n$. Find, but don't solve, the recursion relation for the coefficients.

$f(z) = \sum_{n=0}^{\infty} a_n z^n$

$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1} = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$

$f''(z) = \sum_{n=0}^{\infty} (n)(n+1) a_{n+1} z^{n-1} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} z^n$

$(1-z) f''(z) + 2z f'(z) + 8 f(z) = 0$

$f''(z) - z f''(z) + 2z f'(z) + 8 f(z) = 0$

$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} z^n - \sum_{n=0}^{\infty} (n)(n+1) a_{n+1} z^n + 2\sum_{n=0}^{\infty} n a_n z^n + 8 \sum_{n=0}^{\infty} a_n z^n = 0$

Equating coefficients:

$(n+1)(n+2) a_{n+2} - n(n+1) a_{n+1} + 2n a_n + 8 a_n = 0$

$\boxed{(n+1)(n+2) a_{n+2} - n(n+1) a_{n+1} + (2n+8) a_n = 0}$

✓

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