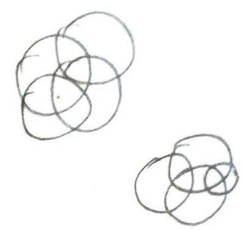
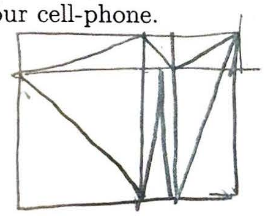
First Name: Jacob ID# [redacted]

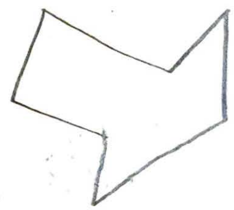
Last Name: Zhang

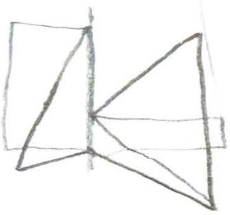


Rules:

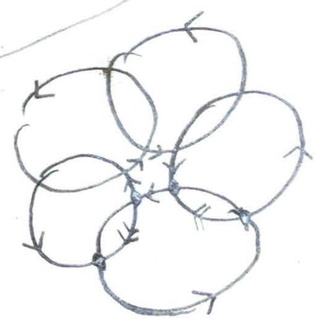
- There are **THREE** problems, each worth 10 points.
- Use the backs of the pages.
- No calculators, computers, notes, books, e.t.c..
- Out of consideration for your classmates, no chewing, humming, pen-twirling, snoring, e.t.c.. Try to sit still.
- Turn off your cell-phone.

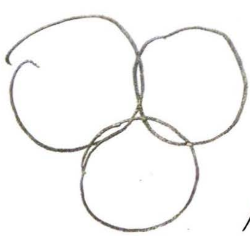


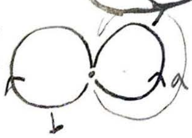




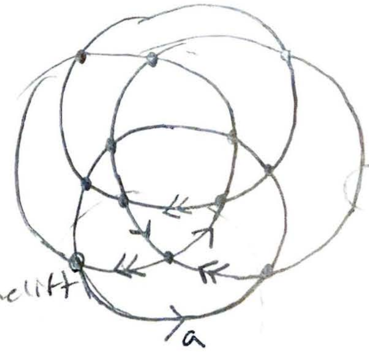
1	2	3	4	Σ
10	10	10		30







X

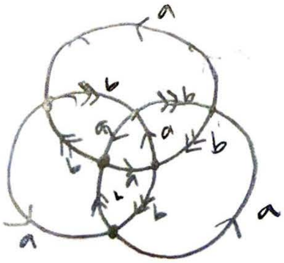


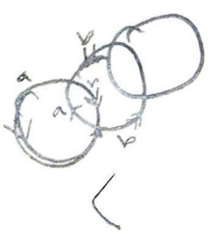
ab has a unique lift

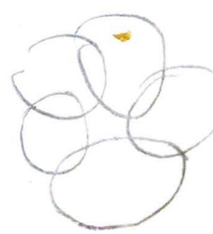
$(ab)^n$

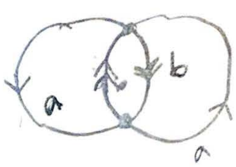
n circles $=4$

12-sheeted cover









$\tilde{X}$

$n=2$ ~~sea~~

2 sheets

generated by

ab^2 $ba^{-2}b$

bab $(ab)^6$

b^2a

Image of $\pi_1(\tilde{X})$ in $\pi_1(X)$ is generated by a^2, b^2, ab, ba = subgroup of even-length words

Problem 1. Let
$$X = \{f : \mathbb{N} \to \{0, 1\}\}$$
denote the space of sequences with values in $\{0, 1\}$. On X we define a metric via
$$d(f, g) = \sum_{n \in \mathbb{N}} 2^{-n} |f(n) - g(n)|.$$
Show that with this metric, the space X is compact.
(You need not show that d is a metric; you may use this fact freely.)

Note: This result is basically König's Lemma / the infinite tree theorem.

We show that X is sequentially compact and therefore compact.

Let $f_1, f_2, f_3, \ldots$ be an infinite sequence of elements of X. Since it is infinite, it either contains an infinite number of functions satisfying $f_i(1) = 0$ or $f_i(1) = 1$. Suppose it has an infinite number of terms f_i with $f_i(1) = b_1$. Let $f_i^{(1)}$ be the subsequence of such terms. Now $f_i^{(1)}$ is infinite, so for some $b_2 \in \{0,1\}$ there is an infinite subsequence such that $f_i^{(1)}(2) = b_2$ for all $f_i^{(1)}$ in the subsequence; call this subsequence $f_i^{(2)}$. Continue this process, obtaining a sequence of sequences $\{\{f_i^{(k)}\}_{i \in \mathbb{N}}\}_{k \in \mathbb{N}}$ with the property that for all i, k and all $j \leq k$ we have $f_i^{(k)}(j) = b_j$.

Now let $g_k = f_k^{(k)}$ be the diagonal subsequence and let $g(n) = b_n \in X$. We claim $\{g_k\}$ converges to g under d. Indeed, since $g_k(n) = g(n)$ for $n \leq k$,

$$d(g, g_k) = \sum_{n \in \mathbb{N}} 2^{-n} |g(n) - g_k(n)|$$
$$= \sum_{n > k} 2^{-n} |g(n) - g_k(n)| \leq \left[\sum_{n > k} 2^{-n} = 2^{-k} \right]^{\dagger}$$

and so for all ε, if we choose N such that $2^{-N} < \varepsilon$, we will have $d(g, g_k) < \varepsilon$ for all $k \geq N$, and so $\{g_k\} \xrightarrow{d} g$. *

* Since $\{g_k\}$ is a subsequence of $\{f_i\}$, we have shown sequential compactness.

† proof: $\sum_{n=k+1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}\right)^{k+1}}{1 - \frac{1}{2}} = 2^{-k}$.

/

Problem 2. Let $f : [1, \infty) \to \mathbb{R}$ be a continuous function such that

$$\lim_{x\to\infty} |f(x)| = 0.$$

For $n \geq 1$, let $g_n : [1, \infty) \to \mathbb{R}$ be given by

$$g_n(x) = f(nx).$$

Show that $\{g_n\}_{n\geq 1}$ is equicontinuous on $[1, \infty)$.

Let ε be given.

Since $\lim_{x\to\infty} |f(x)| = 0$, choose $N \in \mathbb{N}$ so that $|f(x)| < \frac{\varepsilon}{2}$ for all $x \geq N$.

Since the interval $[1, N+1]$ is compact, f is uniformly continuous on the interval. Take δ' such that $|x-y| < \delta' \to |f(x)-f(y)| < \varepsilon$ on $[1, N+1]$, and assume $\delta' < 1$. Now take $\delta = \frac{\delta'}{N}$.

We claim that $|x-y| < \delta \to |g_n(x) - g_n(y)| < \varepsilon$ for all n, x, y. Indeed if ~~[illegible]~~ nx, ny both $\geq N$, covering all the cases where $n \geq N$, then

$$|g_n(x) - g_n(y)| = |f(nx) - f(ny)| \leq |f(nx)| + |f(ny)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

On the other hand if one of $nx, ny < N$, then $|nx - ny| < n\delta = \frac{n}{N}\delta' \leq \delta' < 1$ implies that both $nx, ny \in [1, N+1]$. Then since $|nx - ny| < \delta'$, we have

$$|g_n(x) - g_n(y)| = |f(nx) - f(ny)| < \varepsilon$$

by construction.

Therefore $\forall \varepsilon\ \exists \delta$ such that $\forall n$ $|x-y| < \delta \to |g_n(x) - g_n(y)| < \varepsilon$

So we have shown equicontinuity.

Very clean!

Problem 3.

$X \xrightarrow{f} Y \xrightarrow{g} g(Y)$

compact cts injective bijective ↑ compact

Let (X, d_X), (Y, d_Y) and (Z, d_Z) be three metric spaces such that Y is compact.
Let $f : X \to Y$ and $g : Y \to Z$ be two functions such that g is continuous and injective.
Assume that $g \circ f : X \to Z$ is uniformly continuous. Show that f is uniformly continuous.

Let ε be given.

never used [Since g is a cts function on a compact set, g is uniformly cts.]

The set $g(Y) \subset Z$ is a subspace of Z under d_Z. Since Y compact and g cts, $g(Y)$ is compact. Since $g: Y \to Z$ is injective, $g: Y \to g(Y)$ is a (set) bijection.

We claim that the map between sets $g^{-1}: g(Y) \to Y$ is actually a continuous function. Indeed, we show that the preimage of a closed set under g^{-1} is closed. If $A \subset Y$ is closed, then $(g^{-1})^{-1}(A) = g(A)$ is closed because $A \subset Y$ closed, Y compact $\to$ A compact $\to$ $g(A)$ compact $\to$ $g(A)$ closed.

this equality uses that g is a biject. (obviously).

Now since we have the cts function $g^{-1}: g(Y) \to Y$, with compact domain, we know it is uniformly continuous. Take δ such that $d_Z(z_1, z_2) < \delta \Rightarrow d_Y(g^{-1}(z_1), g^{-1}(z_2)) < \varepsilon$ for $z_1, z_2 \in g(Y)$. Using uniform continuity of $g \circ f$, take δ' such that $d_X(x_1, x_2) < \delta' \Rightarrow d_Z(g(f(x_1)), g(f(x_2))) < \delta$.
Then if $d_X(x_1, x_2) < \delta'$, we have

$$d_Y(f(x_1), f(x_2)) = d_Y(g^{-1}(g(f(x_1))), g^{-1}(g(f(x_2)))) < \varepsilon$$

since $g(f(x_1)), g(f(x_2)) \in g(Y)$ with $d_Z(g(f(x_1)), g(f(x_2))) < \delta$ thus establishing that $f: X \to Y$ uniformly cts.