**Math 131BH - Lecture 1**
**Spring 2021**
**Final exam**
**Due 6/8/2021 before 10am**

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**Sign and submit the following honor statement:**

*I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.*

| **Signed:** |
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| **Print name:** |
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This exam contains 8 pages (including this cover page) and 7 problems. There are a total of 80 points available.

- Include extra pages as you need them.
- Attempt all questions.
- The work submitted must be entirely your own: you may not discuss it with anyone else.
- You may email the instructor `visan@math.ucla.edu` with any queries about what the questions are asking.
- This exam is open book. You may use your notes, the textbook, and any online resource that does not involve interaction with another person.
- **Posting problems to online forums or "tutoring" websites counts as interaction with another person so is strictly forbidden.**

1. (10 points) Let $(X, d)$ be a metric space and let $C$ and $K$ be subsets of $X$ such that $C$ is closed and $K$ is compact. Show that *exactly one* of the following two statements holds:

   (1) $C \cap K \neq \varnothing$

   (2) $\inf\{d(x, y) : x \in K \text{ and } y \in C\} > 0$.

2. (10 points) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Show that the function $g : [0, 1] \to \mathbb{R}$ given by

$$g(x) = \inf\{f(y) + xy : y \in [0, 1]\}$$

is continuous.

3. (10 points) Fix $n \geq 1$ and let $A$ be a non-empty closed subset of $\mathbb{R}^n$ such that

$$\text{if} \quad x, y \in A \quad \text{then} \quad \tfrac{x+y}{2} \in A.$$

Show that $A$ is path connected.

4. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with the derivative $f'$ continuous. Show that

$$\lim_{h\to\infty} h^2 \int_0^1 e^{-ht} \big[ f(t) - f(0) \big] \, dt = f'(0).$$

5. (10 points) Let $f : [0,1] \to \mathbb{R}$ be a function satisfying

$$\sup_{n\in\mathbb{N}} \sup_{0\le t_0<\cdots<t_n\le 1} \sum_{k=1}^{n} \big| f(t_k) - f(t_{k-1}) \big|^2 \le M < \infty.$$

Show that $f$ is Riemann integrable.

6. (15 points) Let $(X,d)$ be a compact metric space and let $\Gamma = \big\{\gamma : [0,1] \to X : \gamma \text{ is continuous}\big\}$. We define the cost function on paths $c : \Gamma \to [0,\infty]$ via

$$c(\gamma) = \sup_{0 \le s \ne t \le 1} \frac{d(\gamma(t),\gamma(s))}{|t-s|}$$

Fix two distinct points $x, y$ in $X$ and assume that they are connected by a path $\gamma_0 \in \Gamma$ of finite cost, that is,

$$\gamma_0(0) = x, \quad \gamma_0(1) = y, \quad \text{and} \quad c(\gamma_0) < \infty.$$

Show that there is a path of minimal cost that connects $x$ and $y$, that is, there exists $\tilde{\gamma} \in \Gamma$ such that

$$\tilde{\gamma}(0) = x, \quad \tilde{\gamma}(1) = y, \quad \text{and} \quad c(\tilde{\gamma}) = \inf\big\{c(\gamma) \,|\, \gamma \in \Gamma \text{ with } \gamma(0) = x \text{ and } \gamma(1) = y\big\}.$$

*Hint:* While the Arzela-Ascoli theorem is not directly applicable, you might want to use the ideas in its proof.

7. (15 points)

(a) Show that

$$|\theta| \leq \tfrac{\pi}{2}|\sin(\theta)| \quad \text{for all} \quad \theta \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

and so deduce that the Fejér kernels $F_N$ satisfy the inequality

$$F_N(x) \leq \frac{1}{4Nx^2} \quad \text{for all } x \in [-\tfrac{1}{2}, \tfrac{1}{2}] \text{ and } N \geq 1.$$

(b) Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous, 1-periodic function. Assume that there exists $\alpha \in (0, 1)$ such that

$$|f(x) - f(y)| \leq |x - y|^\alpha \quad \text{for all} \quad x, y \in \mathbb{R}.$$

Show that for every $N \geq 1$ there exists a trigonometric polynomial

$$P_N(x) = \sum_{|n| \leq N-1} c_n e^{2\pi i n x} \quad \text{with coefficients } c_n \in \mathbb{C}$$

such that

$$\sup_{x \in \mathbb{R}} \left| f(x) - P_N(x) \right| \leq \frac{C}{N^\alpha}$$

for some constant $C > 0$ which is allowed to depend on $\alpha$, but not on $N$.