MATH 131BH: SECOND MIDTERM

SPRING 2022

LAST NAME ____________________________

FIRST NAME ____________________________

ID NO. ____________________________

Instructions: Perform the stated tasks while keeping in mind the following points:

- This is an exam in rigorous mathematics which may appear light on content but that is because the grading will pay a very serious attention to mathematical details.
- All of the problems generally ask for self-contained proofs (either explicitly or by default) so if you want to borrow on extraneous facts (from class or homework or the textbook) you have to state and prove those as well. That being said, it is fine to use well known (and important) facts without proof but you still have to state these precisely with all the needed conditions. The priority is on the main line of the argument so first address that and then proceed to deal with the asides.
- In your proofs, all the objects that you invoke need to be properly defined and all steps in your logical reasoning need to be fully justified.

All these points will be reflected on in the grading. Each problem is worth the same amount of points.

DO NOT WRITE BELOW THIS LINE!

$\boxed{1}$ ____________ $\boxed{2}$ ____________

$\boxed{3}$ ____________ $\boxed{4}$ ____________

TOTAL SCORE ____________

PROBLEM 1

Let $f\colon \mathbb{R} \to \mathbb{R}$ be twice differentiable without f'' necessarily continuous. Do the following:

(1) For all $x \in \mathbb{R}$ and $h > 0$, prove there is $t \in (-h, h)$ such that

$$\frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x+t)$$

(2) Prove that for all $x \in \mathbb{R}$,

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x)$$

(3) Give an example of f such that the limit in (2) exists at some x yet $f''(x)$ does not exist.

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(1)

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(2)

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(3)

PROBLEM 2

Let $a < b$ be reals and $f, g\colon [a,b] \to \mathbb{R}$ functions. Do as follows:

(1) Define the phrase "f is Stietljes integrable with respect to g on $[a,b]$ in Riemann sense" and the meaning of the symbol $\int_a^b f \, \mathrm{d}g$.

(2) Abbreviating the phrase in (1) as "$f \in \mathrm{RS}(g, [a,b])$" prove

$$f \in \mathrm{RS}(g, [a,b]) \;\Rightarrow\; g \in \mathrm{RS}(f, [a,b])$$

and prove that $f \in \mathrm{RS}(g, [a,b])$ implies

$$\int_a^b f \, \mathrm{d}g + \int_a^b g \, \mathrm{d}f = f(b)g(b) - f(a)g(a)$$

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(1)

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(2)

PROBLEM 3

Let $a < b$ be reals and, for each $n \in \mathbb{N}$, let $f_n, F_n \colon [a,b] \to \mathbb{R}$ be functions. Do as follows:

(1) Define the phrase "$\{F_n\}_{n\in\mathbb{N}}$ converges uniformly on $[a,b]$."
(2) Define the phrase "$\{f_n\}_{n\in\mathbb{N}}$ is uniformly bounded on $[a,b]$."
(3) Assuming that

$$\forall x \in [a,b]: \quad F_n(x) := \int_a^x f_n(t)\,\mathrm{d}t$$

for $\{f_n\}_{n\in\mathbb{N}}$ uniformly bounded and with each f_n Riemann integrable, prove that $\{F_n\}_{n\in\mathbb{N}}$ contains a uniformly convergent subsequence.

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(1)

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(2)

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(3)

PROBLEM 4

Consider the functions defined formally by

$$f(x) := \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \ \wedge \ g(x) := \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

Do as follows:

(1) Prove that the series converge for all $x \in \mathbb{R}$.
(2) Prove that both f and g are continuously differentiable on $\mathbb{R}$.
(3) Prove that

$$\forall x \in \mathbb{R}: \ f(x)^2 - g(x)^2 = 1$$

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(1)

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(2)

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(3)

SCRATCH PAPER: Will NOT be looked at