Midterm 1

04/19-20/2021

131B
J. Madrid

Your name: UID:

Instructions: You have 24 **hours** to complete this exam **from Monday April 19 at 8:00 am to Tuesday April 20 at 8:00 am, Pacific time**.

There are 3 problems, worth a total of 25 points. This test is OPEN book and OPEN notes. Calculators are allowed. **Collaboration of any kind is NOT allowed.**

For full credit show all of your work legibly and **justify all your answers!**

Please write your solutions on white papers. After completion, scan your solutions, put all of them in a single file and convert to pdf. You have to submit your pdf file via Gradescope by Tuesday April 20 at 8:00 am PST. You don't need to attach your scratch work.

Please **circle or box your final answers**.

Problem	Points	Score
1	9	
2	8	
3	8	
Total	25	

Honor Statement. It is a **departmental policy** that any remote evaluation should contain the following statement. This statement **must** be signed by the student. **If it is not signed, the evaluation should be given a failing grade**.

I assert, on my honor, that I have not received assistance of any kind from any other person while working on the midterm and that I have not used any non-permitted materials or technologies during the period of this evaluation.

Student signature: ______________________

Problem 1. *9pts.*

Let

$$X := \{(a_n)_{n\geq 1} |\ a_n \in \mathbb{R}\}.$$

Define $d : X \times X \to [0, +\infty)$ by

$$d((a_n)_{n\geq 1}, (b_n)_{n\geq 1}) := \frac{1}{\min\{m \in \mathbb{N} : a_m \neq b_m\}}$$

if $(a_n)_{n\geq 1} \neq (b_n)_{n\geq 1}$, and otherwise

$$d((a_n)_{n\geq 1}, (b_n)_{n\geq 1}) = 0.$$

i.) (2 points) Prove that d defines a metric on X.

ii.) (3 points) Let $A \subset X$ be the set of all sequences $(a_n)_{n\geq 1}$ which EITHER begin with $0, 1, 2$ (namely: $a_1 = 0$ and $a_2 = 1$ and $a_3 = 2$) OR begin with $3, 4, 5, 6$ (namely: $a_1 = 3$ and $a_2 = 4$ and $a_3 = 5$ and $a_4 = 6$). Prove that A is open in (X, d).

iii.) (4 points) Prove that the set of all constant sequences is closed in (X, d).

Problem 2. *8pts.*

i.) (3 points) Let (X, d) be a metric space, and let $K_1, K_2, \ldots, K_n$ be a finite family of compact subsets of X. Proved that $\cup_{i=1}^{n} K_i$ is a compact set.

ii.) (5 points) Let (X, d) be a metric space, let $K \subseteq X$ be compact and let $C \subseteq X$ be closed. Assume that

$$\inf\{d(x, y); x \in K, y \in C\} = 0.$$

Prove that $K \cap C \neq \emptyset$.

Problem 3. *8pts.*

True or False? **Justify your answers!**

In each of the following items a metric space is given (you do NOT need to verify that d is in fact a metric, you can just assume it). For each metric space, you are asked to determine whether or not it is complete, and to *prove your answer.*

i.) (4 points) Let $d : \mathbb{R}^k \times \mathbb{R}^k \to [0, +\infty)$ defined by

$$d((a_1, a_2, \ldots, a_k), (b_1, b_2, \ldots, b_k)) := \max\{|a_n - b_n| : n \in \{1, 2, \ldots, k\}\}.$$

Is the metric space $(\mathbb{R}^k, d)$ complete? Prove your answer!

ii.) (4 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a bijective function (namely: f is both one-to-one and onto). Define $d_f : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ by

$$d_f(x, y) := |f(x) - f(y)|.$$

Is the metric space $(\mathbb{R}, d_f)$ complete? Prove your answer!