Problem 1. (15 points)

(a) Prove that $\sqrt{3}$ is not a rational number.

(b) Prove that the least upper bound of the set $\{x \in \mathbb{Q} : x \leq \sqrt{3}\}$ exists and is $\sqrt{3}$.

(c) Let S be a non-empty subset of the real numbers. Define what it means for S to have the *least upper bound property*.

(d) Does $\mathbb{Q}$ have the least upper bound property? Justify your answer.

4/5 (a) Assume by contradiction that $\sqrt{3}$ is rational.
Then $\exists p,q \in \mathbb{Z}$ s.t. $\frac{p}{q} = \sqrt{3}$, with p,q coprime
$\Rightarrow p^2 = 3q^2$
$\Rightarrow p = 3k$ for some $k \in \mathbb{Z}$
$\Rightarrow (3k)^2 = 3q^2 \Rightarrow 3k^2 = q^2$
$\Rightarrow q = 3l$ for some $l \in \mathbb{Z}$
But then, both p,q have factors of 3, and they are coprime, contradiction.
So $\sqrt{3}$ is not a rational number. ✓

4/6 b) Clearly $\sqrt{3}$ is an upper bound. We will show it is the least upper bound.
Let $M < \sqrt{3}$. Then M is not an upper bound, since as $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists q \in \mathbb{Q}$ s.t. $M < q < \sqrt{3}$. Since this $q \in \{x \in \mathbb{Q} : x \leq \sqrt{3}\}$ and $q > M$, M is not an upper bound. So $\sqrt{3}$ is the least upper bound, and the least upper bound exists. ?

1/2 c) S has the least upper bound property iff, for any subset $A \subseteq S$ with an upper bound, the least upper bound M exists and $M \in S$

1/2 d) $\mathbb{Q}$ does not have the least upper bound property. From (b), the least upper bound of $\{x \in \mathbb{Q} : x \leq \sqrt{3}\}$ is $\sqrt{3}$, but by (a), $\sqrt{3} \notin \mathbb{Q}$, so by (c), $\mathbb{Q}$ does not have the least upper bound property.

Problem 2. (12 points) Let $\{x_n\}_{n\geq 1}$ be a sequence defined by the following rule:

$$x_1 = 1 \quad \text{and} \quad x_{n+1} = \frac{x_n}{2} + 1 \quad \text{for all} \quad n \geq 1.$$

(a) Assuming that $\{x_n\}_{n\geq 1}$ converges to $l \in \mathbb{R}$, find l.
(b) Using induction, prove that l found above is an upper bound for the sequence $\{x_n\}_{n\geq 1}$.
(c) Show that $\{x_n\}_{n\geq 1}$ is monotonically increasing.
(d) Deduce that $\{x_n\}_{n\geq 1}$ converges.

(a) $l = \lim_{n\to\infty} X_n = \lim_{n\to\infty} X_{n+1}$

$X_{n+1} = \frac{X_n}{2} + 1$

$\lim_{n\to\infty} X_{n+1} = \lim_{n\to\infty}\left(\frac{X_n}{2}+1\right) = \frac{1}{2}\lim_{n\to\infty} X_n + 1$

$l = \frac{1}{2}l + 1$

$\frac{1}{2}l = 1$

$l = 2$

(b) Let $P(n)$ be the statement that $X_n < \sqrt{2}$ $\forall n \geq 1$

Step 1: $X_1 = 1 < 2$, so $P(1)$ holds

Step 2: Assume $P(n)$ holds. Then $X_n < 2$

$\Rightarrow X_{n+1} = \frac{X_n}{2} + 1 < \frac{2}{2} + 1 = 2$, so $P(n+1)$ holds

Collecting steps 1 & 2, by induction, $P(n)$ holds $\forall n \geq 1$, so $X_n < 2$ $\forall n \geq 1$

Therefore $l = 2$ is an upper bound for the sequence $\{X_n\}_{n\geq 1}$

(c) $X_{n+1} = \frac{X_n}{2} + 1 = \frac{X_n + 2}{2} > \frac{X_n + X_n}{2} = X_n$ $\forall n \geq 1$

So $X_{n+1} > X_n$ $\forall n \geq 1$, so $\{X_n\}_{n\geq 1}$ is monotonically increasing

(d) As $\{X_n\}_{n\geq 1}$ is bounded above and monotonically increasing, it converges.

Let $L = \sup\{X_n : n \geq 1\}$.

Let $\varepsilon > 0$. Then $L - \varepsilon$ is not an upper bound, so $\exists\, n_\varepsilon \in \mathbb{N}$ s.t. $X_{n_\varepsilon} > L - \varepsilon$

As $\{X_n\}_{n\geq 1}$ is increasing, $X_n > L - \varepsilon$ and $X_n \leq L < L + \varepsilon$ $\forall n \geq n_\varepsilon$

So $L - \varepsilon < X_n < L + \varepsilon \Rightarrow |X_n - L| < \varepsilon$ $\forall n \geq n_\varepsilon$, so by definition, $\{X_n\}_{n\geq 1}$ converges.

Problem 3. (8 points)

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(a) Let a and p be two *positive* real numbers. Prove that the series

$$\sum_{n\geq 1} \frac{a^n}{n^p}$$

converges if $0 < a < 1$ and diverges if $a > 1$.

(b) Assume now that $a = 1$. What are the values of $p > 0$ for which the series converges? No justification is required.

(a) As $a, p > 0$, clearly $\frac{a^n}{n^p}$ is positive $\forall n \geq 1$

So by the ratio test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{a^{n+1}}{a^n}\cdot\frac{n^p}{(n+1)^p}\right| = \lim_{n\to\infty}\left|a\cdot\left(1-\frac{1}{n+1}\right)^p\right| = a\lim_{n\to\infty}\left|\left(1-\frac{1}{n+1}\right)^p\right| = a\cdot 1 = a$$

So if $0<a<1$, then the series converges, and if $a>1$, the series diverges by the ratio test. ✓

(b) The series is then $\sum_{n\geq 1}\frac{1}{n^p}$.

The series converges iff $p>1$ ✓

Problem 4. (10 points) Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers.

(a) Define
$$\liminf_{n\to\infty} a_n$$

(b) Assume
$$\liminf_{n\to\infty} a_n = 0.$$

Prove that there exists a subsequence of $\{a_n\}_{n\geq 1}$ that converges to 0.

(a) Let S be a set of real numbers.
Define $\inf S$ to be the greatest lower bound of S, and
$\inf S = -\infty$ if S has no lower bound

Then $\liminf_{n\to\infty} a_n = \lim_{N\to\infty}\left(\inf\{a_n : n \geq N\}\right)$, and
$\liminf_{n\to\infty} a_n = -\infty$ if $\{a_n\}_{n\geq 1}$ has no lower bound

(b) We will construct such a subsequence $\{a_{k_n}\}_{n\geq 1}$

As $\liminf_{n\to\infty} a_n = 0$, $\exists N_1 \in \mathbb{N}$ s.t. $|\inf\{a_n : n\geq N\} - 0| < 1$ $\forall N \geq N_1$

In particular, $|\inf\{a_n : n \geq N_1\}| < 1$
$\Rightarrow -1 < \inf\{a_n : n\geq N_1\} < 1$

So 1 is not a lower bound of $\{a_n : n \geq N_1\}$, so $\exists k_1 \in \mathbb{N}$, $k_1 \geq N_1$,
Such that $a_{k_1} < 1$. So $-1 < \inf\{a_n : n \geq N_1\} \leq a_{k_1} < 1$
$\Rightarrow |a_{k_1}| < 1$

We proceed inductively to construct the rest of the subsequence
Assume we have $a_{k_1}, a_{k_2}, \ldots a_{k_{i-1}}$, with $|a_{k_j}| < \frac{1}{j}$ $\forall$ $1 \leq j \leq i-1$
We will construct a_{k_i}

As $\liminf_{n\to\infty} a_n = 0$, $\exists N_i \in \mathbb{N}$ s.t. $|\inf\{a_n : n \geq N\} - 0| < \frac{1}{i}$ $\forall N \geq N_i$

Let $N_1 = \max\{N_i, k_{i-1}+1\}$

In particular, $|\inf\{a_n : n \geq N_1\}| < \frac{1}{i} \Rightarrow -\frac{1}{i} < \inf\{a_n : n\geq N_1\} < \frac{1}{i}$

So $\frac{1}{i}$ is not a lower bound of $\{a_n : n \geq N_i\}$,

So $\exists k_i \in \mathbb{N}$, $k_i \geq N_i$, such that $a_{k_i} < \frac{1}{i}$

So $-\frac{1}{i} < \inf\{a_n : n \geq N_i\} \leq a_{k_i} < \frac{1}{i}$

$\Rightarrow |a_{k_i}| < \frac{1}{i}$

Since $k_i \geq N_i = \max(N_i, k_{i-1}+1)$, $k_i > k_{i-1}$

By induction, we have created a subsequence $\{a_{k_n}\}_{n \geq 1}$ that converges to 0.

- For any $\varepsilon > 0$, by the Archimedean principle, $\exists N \in \mathbb{N}$ s.t.

$N > \frac{1}{\varepsilon} \Rightarrow \frac{1}{N} < \varepsilon$

$\forall n \geq N$, $|a_{k_n}| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$,

So $\{a_{k_n}\}_{n \geq 1}$ converges to 0.

very good!