131AH Visan Midterm November 8th

First Name: Jin Jing FOR COMMENT

Last Name: Zhou

Rules:

- There are **FOUR** problems.
- Use the backs of the pages.
- No calculators, computers, notes, books, e.t.c..
- Out of consideration for your classmates, no chewing, humming, pen-twirling, snoring, e.t.c.. Try to sit still.
- Turn off your cell-phone.

1	2	3	4	Σ
11	12	7	8	38

Problem 1. (15 points)
(a) Prove that $\sqrt{3}$ is not a rational number.
(b) Prove that the least upper bound of the set $\{x \in \mathbb{Q} : x \le \sqrt{3}\}$ exists and is $\sqrt{3}$.
(c) Let S be a non-empty subset of the real numbers. Define what it means for S to have the *least upper bound property*.
(d) Does $\mathbb{Q}$ have the least upper bound property? Justify your answer.

4/5 (a) Assume $\sqrt{3}$ is a rational number $\Rightarrow$ it can be represented as $\frac{a}{b}$ where $a, b \in \mathbb{Z}$, and $\gcd(a,b) = 1$.

Then $(\sqrt{3})^2 = (\frac{b}{a})^2$

$3 = \frac{b^2}{a^2}$

?

$b^2 = 3a^2$ Since $a, b \in \mathbb{Z}$, b must be divisible by 3, let $b = 3k$ where $k \in \mathbb{Z}$

$(3k)^2 = 3a^2$

$3a^2 = 9k^2$

$a^2 = 3k^2$ By the same reasoning, a must be divisible by 3

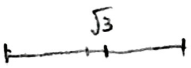
$\Rightarrow \gcd(a,b) \geq 3 \Rightarrow$ contradiction!

Thus, $\sqrt{3}$ is not a rational number.

5/6 (b) Since $\mathbb{R}$ has the least upper bound property, the least upper bound of the set must exist in $\mathbb{R}$. why?

Claim $\sqrt{3}$ is an upper bound.

By definition, $\forall x \in \{x \in \mathbb{Q} : x \le \sqrt{3}\}$, $x \le \sqrt{3}$. $\Rightarrow \sqrt{3}$ is an upper bound



claim $\sqrt{3}$ is ~~an~~ least upper bound

Let $M \in \mathbb{R}$ be another upper bound. Then $\forall x \in \{x \in \mathbb{Q} : x \le \sqrt{3}\}$, $x \le M$.

Assume $M < \sqrt{3}$. By density of $\mathbb{Q}$ on $\mathbb{R}$, $\exists a \in \mathbb{Q}$ such that $M < a < \sqrt{3}$.

Since $a \in \mathbb{Q}$ and $a < \sqrt{3}$, $a \in \{x \in \mathbb{Q} : x \le \sqrt{3}\}$ and $a > M$

$\Rightarrow M$ is not an upper bound

$\Rightarrow$ contradiction $\Rightarrow \sqrt{3}$ is ~~an~~ least upper bound.

?

1/2 (c) S has the least upper bound property if and only if for all subsets of S, there exist a least upper bound that belongs to S.

1/2 (d) $\mathbb{Q}$ does not have the least upper bound property.

A counter-example would be the set in (b) which is a subset of $\mathbb{Q}$, but its least upper bound $\sqrt{3} \notin \mathbb{Q}$.

Problem 2. (12 points) Let $\{x_n\}_{n\geq 1}$ be a sequence defined by the following rule:

$$x_1 = 1 \quad \text{and} \quad x_{n+1} = \frac{x_n}{2} + 1 \quad \text{for all} \quad n \geq 1.$$

(a) Assuming that $\{x_n\}_{n\geq 1}$ converges to $l \in \mathbb{R}$, find l.

(b) Using induction, prove that l found above is an upper bound for the sequence $\{x_n\}_{n\geq 1}$.

(c) Show that $\{x_n\}_{n\geq 1}$ is monotonically increasing.

(d) Deduce that $\{x_n\}_{n\geq 1}$ converges.

(a) $\lim_{n\to\infty} x_n = l$ By uniqueness of limit, $\lim_{n\to\infty} x_{n+1} = l$.

$$x_{n+1} = \frac{x_n}{2} + 1$$

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{x_n}{2} + 1$$

$$\lim_{n\to\infty} x_{n+1} = \frac{\lim_{n\to\infty} x_n}{2} + 1 \qquad \text{since } \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = l$$

$$l = \frac{l}{2} + 1$$

$$l = 2$$

(b) Let P_n be the proposition that $x_n \leq l, \forall n \in \mathbb{N}$.

Base case $x_1 = 1 < 2$ so P_1 is true.

Inductive step Assume for some $k \in \mathbb{N}$, P_k is true, then need to prove P_{k+1} is true.

$$x_{k+1} = \frac{x_k}{2} + 1 \quad \text{since } x_k \leq 2 \text{ (P_k is true)}$$

$$\leq \frac{2}{2} + 1 = 2 \quad \Rightarrow x_{k+1} \leq 2 \Rightarrow P_{k+1} \text{ is true}$$

By principle of mathematical induction, P_n is true $\forall n \in \mathbb{N}$.

Since $2 \geq x_n$ $\forall n \in \mathbb{N}$, 2 is an upper bound for $\{x_n\}_{n\geq 1}$

(c) $x_{n+1} - x_n = (\frac{x_n}{2} + 1) - x_n = 1 - \frac{x_n}{2}$ since $x_n \leq 2$ $\forall n \in \mathbb{N}$.

$$\geq 1 - \frac{2}{2} = 0$$

Thus $x_{n+1} - x_n \geq 0$ $\forall n \in \mathbb{N}$ $\Rightarrow \{x_n\}_{n\geq 1}$ is monotonically increasing.

(d) Since $\{x_n\}_{n\geq 1}$ is monotonically increasing and bounded above by 2, it converges.

Problem 3. (8 points)

(a) Let a and p be two *positive* real numbers. Prove that the series

$$\sum_{n\geq 1} \frac{a^n}{n^p}$$

$\sum \frac{1}{n^p}$

converges if $0 < a < 1$ and diverges if $a > 1$.

(b) Assume now that $a = 1$. What are the values of $p > 0$ for which the series converges? No justification is required.

5/6

(a) If $0<a<1$,

$\sum_{n\geq 1} a^n$ converges (to $\frac{1}{1-a} - 1$) since it is a geometric series

Claim $\frac{1}{n^p}$ is bounded.

Since p is positive, $n^p \geq n$ $\forall p>0, n\geq 1$

$\frac{1}{n^p} \leq \frac{1}{n}$ since $n\geq 1$, $\frac{1}{n} \leq 1$

≤ 1 so $\frac{1}{n^p}$ is bounded above by 1

Since n is positive, $\frac{1}{n^p}$ is bounded below by 0.

By Abel Theorem, since $\sum_{n\geq 1} a^n$ converges and $\frac{1}{n^p}$ is bounded, $\sum_{n\geq 1} \frac{a^n}{n^p}$ converges.

If $a>1$,

$\lim_{n\to\infty} a^n = \infty$, as proven before, $\frac{1}{n^p}$ is bounded and $0 \leq \frac{1}{n^p} \leq 1$ $\forall n \in \mathbb{N}$

why?

$\Rightarrow \lim_{n\to\infty} \frac{a^n}{n^p} = \infty$ If $\sum_{n\geq 1} \frac{a^n}{n^p}$ converges, $\lim_{n\to\infty} \frac{a^n}{n^p} = 0$.

Thus $\sum_{n\geq 1} \frac{a^n}{n^p}$ diverges.

2/2

(b) The series converge $\forall p>1$.

Problem 4. (10 points) Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers.
(a) Define
$$\liminf_{n\to\infty} a_n$$
(b) Assume
$$\liminf_{n\to\infty} a_n = 0.$$
Prove that there exists a subsequence of $\{a_n\}_{n\geq 1}$ that converges to 0.

(a) $\liminf_{n\to\infty} a_n = \lim_{N\to\infty} \inf\{a_n : n \geq N\} = \sup_{N\in\mathbb{N}} \inf\{a_n : n\geq N, N\in\mathbb{N}\}$.

-1

if $\{a_n\}$ is bounded below

$\liminf a_n = -\infty$ otherwise

(b) Let $V_N = \inf\{a_n : n\geq N\}\ \forall N\in\mathbb{N}$.

$\lim_{N\to\infty} V_N = 0$.

By definition of limit, $\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ such that $|V_N - 0| < \varepsilon$ $\forall$

$\Rightarrow |V_N| < \varepsilon \quad \forall N \geq N_\varepsilon$.

Take $\varepsilon = 1$, $\exists N_1 \in \mathbb{N}$ such that $|V_N| < 1 \quad \forall N \geq N_1$

$-1 < V_{N_1} \leq 1$ since $V_{N_1} = \inf\{a_n : n\geq N_1\} \leq a_n \quad \forall n \geq N_1$

Let $\varepsilon_1 = \frac{1 - V_{N_1}}{2}$,

$\exists k_1 \geq N_1$ s.t. $V_{N_1} + \varepsilon_1 > a_{k_1} \geq V_{N_1}$, where k_1 is the smallest number satisfying this condition (or otherwise $V_{N_1} + \varepsilon = \inf\{a_n : n\geq N_1\}$)

$\Rightarrow -1 < V_{N_1} \leq a_{k_1} < V_{N_1} + \varepsilon_1 = V_{N_1} + \frac{1 - V_{N_1}}{2} = \frac{1}{2} + \frac{V_{N_1}}{2} < \frac{1}{2} + \frac{1}{2} = 1$

$\Rightarrow -1 < a_{k_1} < 1$.

Then, take $\varepsilon = \frac{1}{2}$, $\exists N_2 \in \mathbb{N}$ such that $|V_N| < \frac{1}{2} \quad \forall N \geq N_2$.

$-\frac{1}{2} < V_{N_2} < \frac{1}{2}$ since $V_{N_2} = \inf\{a_n : n\geq N_2\} \leq a_n \quad \forall n \geq N_2$

Let $\varepsilon_2 = \frac{\frac{1}{2} - V_{N_2}}{2}$, $\exists k_2 \geq N_2$ and $k_2 \geq k_1 + 1$ and k_2 is the smallest number satisfying this condition

why? -1

s.t. $V_{N_2} + \varepsilon_2 > a_{k_2} \geq V_{N_2}$

$\Rightarrow -\frac{1}{2} < V_{N_2} \leq a_{k_2} < V_{N_2} + \varepsilon_2 = V_{N_2} + \frac{\frac{1}{2} - V_{N_2}}{2} = \frac{1}{4} + \frac{V_{N_2}}{2} < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$\Rightarrow -\frac{1}{2} < a_{k_2} < \frac{1}{2}$

Proceed inductively, we will be able to obtain a subsequence $\{a_{k_n}\}_{n\geq 1}$ where $|a_{k_n}| < \frac{1}{n}$ and $\lim_{n\to\infty} a_{k_n} = 0$.