Math 131AH, Honors Analysis, UCLA
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There are 9 problems.

No electronics are permitted. You can use results from the course in your proofs, but please say what results you are using.

(1) Let f and g be real-valued functions on an open interval containing a point x in $\mathbf{R}$. Suppose that f and g are differentiable at x, that $g'(x) \neq 0$, and that $f(x) = g(x) = 0$. Show that

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

(2) Let f be a real-valued function on a closed interval $[a, b]$ such that the second derivative f'' is defined and < 0 everywhere on $[a, b]$. Show that there is a unique point in $[a, b]$ at which f attains its maximum.

What can you say about the order of the set S of points at which f attains its minimum? (That is, give examples to show the possible sizes of S, and prove that you have listed all the possible sizes.)

(3) Let k be a positive integer. Let $E_1 \supset E_2 \supset \cdots$ be a decreasing sequence of nonempty closed bounded subsets of $\mathbf{R}^k$. Show that the intersection $\cap_{n=1}^{\infty} E_n$ is nonempty.

Is that conclusion true if we omit the assumption that the sets E_n are bounded? Prove or give a counterexample.

(4) Let X be a connected metric space. Show that X has property (*): for any $a > 0$ and any points $x, y \in X$, there is a finite sequence of points $p_0 = x$, $p_1, p_2, \ldots, p_n = y$ (for some $n \in \mathbf{Z}^+$) such that $d(p_i, p_{i+1}) < a$ for all $i \in \{0, \ldots, n-1\}$.

Conversely, if X is a nonempty closed subset of $\mathbf{R}^2$ with property (*), is X connected? Prove or give a counterexample.

(5) Does the series $\sum_{n=1}^{\infty}(\sqrt[n]{2}-1)$ converge? (Hint: to get an idea of how close $\sqrt[n]{2}$ is to 1, look at $\log(\sqrt[n]{2})$.) Justify your answer. (You can use what you know about $\log x$ and its inverse function e^x, for example that $\log x$ is defined and continuous on $(0, \infty)$.)

$\log \sqrt[n]{2} = \frac{1}{n}\log 2$

$\sum e^{\frac{1}{n}\log 2}$

$\cancel{\sum 2 \cdot e^{\frac{1}{n}}}$ $\quad \sum_1 (\sqrt[n]{2}-1)$ $\quad (\sum \sqrt[n]{2}) - n.$

$2 \cdot \sum e^{\frac{1}{n}} - n$ $\quad e^{\frac{1}{n}\ln 2}$ $\quad 2$

(6) Write 0 for the origin $(0,0)$ in $\mathbf{R}^2$. Let $f : \mathbf{R}^2 - \{0\} \to \mathbf{R}$ be a continuous function. What can you say about the image of f?

Also, give examples of continuous functions $f : \mathbf{R}^2 - \{0\} \to \mathbf{R}$ whose image is (a) $(0,1)$, (b) $[0,1)$, and (c) $[0,1]$. Justify your answers.

(7) Let X be a compact metric space. Show that X has property (*): for every $a > 0$, there is a finite subset $S = \{x_1, \ldots, x_n\}$ of X (for some $n \in \mathbf{Z}^+$) such that every point in X has distance $< a$ from at least one of $x_1, \ldots, x_n$.

Conversely, if X is a metric space with property (*), is X necessarily compact? What if X is a complete metric space with property (*): is X compact? Prove or give counterexamples.

(8) Let a be a real number > 1. Define a sequence $(b_n)_{n \in \mathbf{N}}$ by $b_0 = 1$ and

$$b_{n+1} = \frac{1}{2}\left(b_n + \frac{a}{b_n}\right)$$

for $n \geq 0$. Show that $\lim_{n\to\infty} b_n = \sqrt{a}$.

(9) You may use that the natural logarithm $f(x) = \log x$ is a differentiable function on $(0,\infty)$ with $f(1) = 0$ and $f'(x) = 1/x$. Show that

$$\log x = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{(x-1)^j}{j}$$

for all $x \in (0,2)$. Does this formula remain true when $x > 2$? Justify your answer.

$\log x = \log 1 = 0$

$\log''x = -\frac{1}{x^2} = -1$

$\log'''x = \frac{1}{2}\frac{1}{x^3} = \frac{1}{2}$

$\log(a+h) = \log a + \frac{\log' a}{1!} h + \frac{\log'' a}{2!} \cdot h^2$

$\log''''x = -\frac{1}{6}\cdot\frac{1}{x^4} = -\frac{1}{6}$

$\frac{1}{24}$.

$a = 1 \quad (x-1) = h$

$\log \quad 0 + 1\cdot(x-1) + -$

$\frac{1}{x} \quad \log''x \quad \left(\frac{1}{x}\right)' \quad -1\cdot\frac{1}{x^2}$

$\log 1+(x-1) = \log 1 + \log' 1\cdot(x-1) + \frac{\log'' 1\,(x-1)^2}{2!} + \frac{\log''' 1\,(x-1)^3}{3!}$

$\frac{1}{2}\cdot\frac{1}{x^3}$