MATH 131A-2 MIDTERM 01 SOLUTION

Exercise 1. $(3+3+4=10$ pts) Write down the sup and inf for the following sets.
(1) $\{(-2)^n \mid n \in \mathbb{Z}\}$, (2) $\{x \mid x^2 < 3\}$, (3) $\{(-1)^n + \frac{1}{n} \mid n \in \mathbb{N}\}$.

Proof. (1) inf $= -\infty$ and sup $= +\infty$.
(2) inf $= -\sqrt{3}$ and sup $= \sqrt{3}$.
(3) inf $= -1$ and sup $= \frac{3}{2}$.
For (3), we see that both $((-1)^{2k} + \frac{1}{2k})$ and $((-1)^{2k+1} + \frac{1}{2k+1})$ are decreasing. Moreover, $\lim_{n\to+\infty} \frac{1}{n} = 0$. □

Exercise 2. $(10+10=20$ pts)Consider the sequence (s_n) such that $s_1 = 1$ and $s_{n+1} = \sqrt{s_n + 2}$.
(1) Show by induction that $2 \geqslant s_{n+1} \geqslant s_n$ for all n.
(2) Show that (s_n) converges to 2.

Proof. (1) (5 pts for induction redaction, 5 pts for calculation)
We will prove by induction on n that

$$2 \geqslant s_{n+1} \geqslant s_n$$

is true for all $n \in \mathbb{N}$. If $n = 1$, then $s_2 = \sqrt{3}$ and hence $2 \geqslant s_2 \geqslant s_1$ is true. Assume that this property is true for some $n \geqslant 1$.

We will now prove that $2 \geqslant s_{n+2} \geqslant s_{n+1}$. Since $s_{n+1} \geqslant s_n$ by induction, we have $\sqrt{s_{n+1}+2} \geqslant \sqrt{s_n + 2}$, which means that $s_{n+2} \geqslant s_{n+1}$. Moreover, since $s_{n+1} \leqslant 2$ by induction, we also have $s_{n+2} = \sqrt{s_{n+1}+2} \leqslant \sqrt{2+2} = 2$. This completes the induction.

Remark: In this exercise, the function $x \mapsto \sqrt{x+2}$ is obviously increasing. Hence we may say, for example, "$\sqrt{s_{n+1}+2} \geqslant \sqrt{s_n+2}$ since $s_{n+1} \geqslant s_n$" without further explanation.

(2)(5 pts for existence of limit, 5 pts for computation)
Since (s_n) is increasing and bounded above by 2 after part (1), the sequence (s_n) converges to some real number s. Thus, by limit theorem, we have $s = \sqrt{s+2}$, which implies that $s^2 - s - 2 = 0$. Thus s has two possible values: 2 and -1. Since $s_n \geqslant 0$ from the definition, we have $s \geqslant 0$. Thus $s = 2$.

Remark: Some people say that "since (s_n) is increasing and bounded above by 2, it converges to 2". This argument is completely false. In fact, in this exercise, (s_n) is increasing and also bounded above by 3, but $\lim s_n$ is not 3. □

Exercise 3. $(8+8+8+6=30$ pts) Consider the sequence (s_n) such that $s_1 = 4$ and $s_{n+1} = \frac{1}{2}(\frac{4}{s_n} + s_n)$ for all positive integer n.
(1) Show that, for all n,

$$s_{n+1}^2 - 4 = \frac{(s_n^2 - 4)^2}{4s_n^2}.$$

(2) Show by induction on n that $s_n > 2$ for all n.

(3) If $a_n = s_n^2 - 4$, then show that $|\frac{a_{n+1}}{a_n}| \leqslant \frac{1}{4}$.
(4) What is the limit of (a_n), and what about the limit of (s_n)?

Proof. (1) For each n, we have

$$\begin{aligned} s_{n+1}^2 - 4 &= \frac{1}{4}(\frac{4+s_n^2}{s_n})^2 - 4 = \frac{16+8s_n^2+s_n^4}{4s_n^2} - 4 \\ &= \frac{16+8s_n^2+s_n^4-16s_n^2}{4s_n^2} = \frac{16-8s_n^2+n^4}{4s_n^2} \\ &= \frac{(s_n^2-4)^2}{4s_n^2}. \end{aligned}$$

(2)(4 pts for induction redaction, 4 pts for computation)
We will show by induction on n that $s_n > 2$ for all $n \geqslant 1$. If $n = 1$, then $s_1 = 4$, which is larger than 2. We assume that $s_n > 2$ for some $n \geqslant 1$.
We will now show that $s_{n+1} > 2$. Since $s_n > 0$, we have $s_{n+1} > 0$. Since $s_n > 2$, we have $s_n^2 - 4 > 0$. Thus $(s_n^2 - 4)^2 > 0$ and hence $s_{n+1}^2 - 4 = \frac{(s_n^2-4)^2}{4s_n^2} > 0$. Since $s_{n+1} > 0$, this shows that $s_{n+1} > 2$ and completes the induction.
Remark: If one just replaces s_n by 2 in the expression of $s_{n+1}^2 - 4$ and claims that $s_{n+1^2} - 4 > \frac{2^2-4}{4\cdot 2^2} = 0$, without mentioning the monotony of the function $x \mapsto \frac{(x-4)^2}{4x}$, then he/she will not get the 4 pts for computation. Also, the formula "$x + \frac{4}{x} \geqslant 4$ for all positive x" does not suffice to conclude the strict inequality $>$. If one uses this method, he/she should mention that the equality of $x + \frac{4}{x} \geqslant 4$ holds if and only if $x = 2$.

(3) Since $a_n = s_n^2 - 4$ for all n, we have $a_n > 0$ and

$$|\frac{a_{n+1}}{a_n}| = |\frac{s_{n+1}^2 - 4}{s_n^2 - 4}| = |\frac{s_n^2-4}{4s_n^2}| = \frac{s_n^2-4}{4s_n^2} = \frac{1}{4} - \frac{1}{s_n^2}.$$

Since $\frac{1}{s_n^2} > 0$, we have $|\frac{a_{n+1}}{a_n}| \leqslant \frac{1}{4}$.

(4) (3 pts for existence of limit, 3 pts for computation)
Since

$$|\frac{a_{n+1}}{a_n}| \leqslant \frac{1}{4},$$

we have $\lim a_n = 0$. Note that $s_n^2 = 4 + a_n$ and $s_n > 0$. We have $s_n = \sqrt{4 + a_n}$. Thus, by limit theorem, $\lim s_n = \sqrt{4+0} = 2$.
Remark: One may assume $\lim s_n = s$ and use limit theorem to show that

$$s^2 - 4 = \frac{(s^2-4)^2}{4s^2}.$$

However, in order to do this, one must show that s exists and is not zero. □

Exercise 4. $(8 + 8 + 8 + 8 + 8 = 40$ pts) Consider the sequence (s_n) such that $s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$.
(1) Show by induction that $s_n \leqslant 3 - \frac{1}{n}$ for every n.
(2) Prove that (s_n) converges.
Let e be the limit of s_n.

(3) Let $t_n = s_n + \frac{1}{n!}$. What is the limit of (t_n)?

(4) Show that $t_n > t_{n+1}$ for all $n \geqslant 2$. Conclude that $s_n < e < t_n$ for all n.

(5) Show that e is irrational. (Hint: assume by contradiction that $e = \frac{p}{q}$ with $q > 0$, what can we say about $q! \cdot s_q$, $q! \cdot e$ and $q! \cdot t_q$?)

Proof. (1) (4 pts for induction redaction, 4 pts for calculation)

We will prove by induction on n that $s_n \leqslant 3 - \frac{1}{n}$ for every n. If $n = 1$, then $s_1 = 2$ and $3 - 1 = 2$. Thus $s_1 \leqslant 2$ is true. Assume that this property holds for some $n \geqslant 1$.

We will now prove that $s_{n+1} \leqslant 3 - \frac{1}{n+1}$. Note that $s_{n+1} = s_n + \frac{1}{(n+1)!}$, thus by induction,

$$s_{n+1} \leqslant 3 - \frac{1}{n} + \frac{1}{(n+1)!}.$$

Since $n \geqslant 1$, we have $(n+1)! \geqslant n(n+1)$. Thus $\frac{1}{(n+1)!} \leqslant \frac{1}{n(n+1)}$. Hence

$$s_{n+1} \leqslant 3 - \frac{1}{n} + \frac{1}{n(n+1)} = 3 - \frac{1}{n+1}$$

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Remark: The idea behind the proof is as follows. It is not hard to achieve $s_{n+1} \leqslant 3 - \frac{1}{n} + \frac{1}{(n+1)!}$. Then it is enough to show that

$$-\frac{1}{n} + \frac{1}{(n+1)} \leqslant -\frac{1}{n+1},$$

which is equivalent to

$$\frac{1}{(n+1)!} \leqslant \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}.$$

This is true since

$$(n+1)! \geqslant n(n+1).$$

This is why we throw out the statement "since $n \geqslant 1$, we have $(n+1)! \geqslant n(n+1)$" in the proof.

(2) From the definition, we see that $s_n < s_{n+1}$ for all $n \geqslant 1$. By part (1), (s_n) is bounded above by 3. Thus (s_n) converges.

Remark: The bound of a sequence should be a constant, independent of n. Thus $3 - \frac{1}{n}$ is not an upper bound of (s_n).

(3) By limit theorem, we have $\lim t_n = \lim s_n + \lim \frac{1}{n!} = e + 0 = e$.

(4) Let $n \geqslant 2$ fixed. Then

$$\begin{aligned} t_{n+1} - t_n &= s_{n+1} + \frac{1}{(n+1)!} - s_n - \frac{1}{n!} = \frac{1}{(n+1)!} + \frac{1}{(n+1)!} - \frac{1}{n} \\ &= \frac{2}{(n+1)!} - \frac{1}{n!} = \frac{1}{n!}(\frac{2}{n+1} - 1). \end{aligned}$$

Since $n \geqslant 2$, we have $2 < n + 1$. Thus $\frac{2}{n+1} - 1 < 0$. Since $\frac{1}{n!} > 0$, we have

$$t_{n+1} - t_n = \frac{1}{n!}(\frac{2}{n+1} - 1) < 0,$$

which is $t_{n+1} < t_n$.

Since (s_n) increases strictly and converges to e, we have $s_n < e$ for all $n \geqslant 1$. Similarly, since (t_n) decreases strictly from term 2, and (t_n) converges to e, we have $e < t_n$ for all n.

Remark: For $s_n < e < t_n$, one must mention that (s_n) is *strictly* increasing and (t_n) is *strictly* decreasing from term 2.

(5) Assume the opposite. Then there is a positive integer q and an integer p such that $e = \frac{p}{q}$. Then $q!e = q! \cdot \frac{p}{q} = (q-1)! \cdot p$ is an integer. (we use the convention that $0! = 1$).

Moreover, $q!s_q = q!(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots \frac{1}{q!})$. Since $q!$ is divided by $i!$ for all positive integer $i \leqslant q$, we obtain that $q!s_q$ is an integer. Thus $q!t_q = q!s_q + 1$ is also an integer. From part (4), we have

$$q!s_q < q!e < q!t_q = q!s_q + 1.$$

Since $q!s_q$ and $q!e$ are integers, this is a contradiction. $\square$